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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON THE COMPLETENESS OF THE SYSTEM OF FINITE-DIMENSIONAL INVARIANT SUBSPACES OF A NONSELFADJOINT SINGULAR DIFFERENTIAL OPERATOR OF SECOND ORDER WITH RESPECT TO A REAL WEIGHT FUNCTION**

*(Presented by Academician L. S. Pontryagin, 17 III 1964)*

In the present paper we consider the question of the completeness of the linear span of the finite-dimensional invariant subspaces of a pair of operators  $(L, Q)$ , connected with the equation

$$-y'' + p(x)y = \lambda q(x)y \tag{1}$$

and acting in  $L^2(-\infty, +\infty)$ . In equation (1),  $p(x) = p_1(x) + ip_2(x)$  is a complex-valued function of the real parameter  $x$ ,  $p_1(x)$  and  $p_2(x)$  are summable on every finite interval of the real axis, and  $q(x)$  is a complex essentially bounded function. We shall assume that the condition

$$\operatorname{Re} p(x) = p_1(x) \geq 1$$

is satisfied.

Under this condition the operator  $L_0$ , generated by the differential expression

$$l(y) = -y'' + (p_1(x) + ip_2(x))y$$

and defined on the manifold  $D_0$  of finite functions  $\mathcal{L}(x)$ , each of which is absolutely continuous together with its derivative and for which  $l(z) \in L^2(-\infty, +\infty)$ , admits the closure  $\overline{L_0} = L$ .

The domain of definition  $D(L)$  of the operator  $L$  is determined by the conditions:  
 a)  $y(x) \in L^2(-\infty, +\infty)$ ,  $y(x)$  and  $y'(x)$  are absolutely continuous on every finite interval of the real axis;  
 b) the function  $-y'' + p(x)y$  belongs to  $L^2(-\infty, +\infty)$ .

The following proposition holds (see ([1])):

1°. Under the conditions  $p_1(x) \rightarrow \infty$  and  $p_1(x) \geq 1$ , the operator  $L$  possesses a completely continuous inverse.

We denote by  $Q$  the operator of multiplication by the essentially bounded function  $q(x)$ , bounded and defined everywhere in  $L^2(-\infty, +\infty)$ .

The number  $\lambda$  will be called an eigenvalue of the pair of operators  $(L, Q)$  if in  $D(L)$  there exists a function  $y \neq 0$  such that  $Ly = \lambda Qy$ ; in this case  $y$  is called an eigenfunction of  $(L, Q)$  corresponding to the eigenvalue  $\lambda$ .

The value  $\lambda$  is called a regular point of the pair of operators  $(L, Q)$  if the operator  $(L - \lambda Q)^{-1}$  exists, is defined on all of  $L^2(-\infty, +\infty)$ , and is bounded. All nonregular points are called points of the spectrum of the pair of operators  $(L, Q)$ .

It is not difficult to show that the spectrum of the pair of operators  $(L, Q)$  consists of no more than a countable number of eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s, \dots$  and is connected with the spectrum of the completely continuous operator  $QL^{-1}$ ,  $\lambda'_1, \lambda'_2, \dots, \lambda'_s, \dots$ , by the equalities  $\lambda_s = 1/\lambda'_s$ .

A subspace  $\mathfrak{N}$  of the space  $L^2$ , belonging to  $D(L)$ , is called an invariant subspace of the pair of opera-

$$\text{tor}(L, Q),$$

if  $Q\mathfrak{N} \subset L\mathfrak{N}$ , i.e., if for every element  $f \in \mathfrak{N}$  there is an element  $g \in \mathfrak{N}$  such that

$$Qf = Lg.$$

One says that  $\mathfrak{N}$  is an invariant subspace corresponding to the eigenvalue  $\lambda$  if  $\mathfrak{N}$  contains no eigenfunctions of the pair of operators  $(L, Q)$  corresponding to eigenvalues different from  $\lambda$ .

If there is no invariant subspace  $\mathfrak{N}_1$  of the pair of operators  $(L, Q)$  such that  $\mathfrak{N}_1 \supset \mathfrak{N}$ , and corresponding to the same eigenvalue as  $\mathfrak{N}_1$ , then  $\mathfrak{N}$  is called a maximal invariant subspace of the pair of operators  $(L, Q)$ , corresponding to the eigenvalue  $\lambda$  (5).

Consider the completely continuous operator  $S = L^{-1}Q$ . Any invariant subspace of the pair of operators  $(L, Q)$  corresponding to the eigenvalue  $\lambda_k$  is an invariant subspace of the operator  $S$ , corresponding to the eigenvalue  $\lambda'_k = 1/\lambda_k$ . Therefore, to each eigenvalue of the pair of operators  $(L, Q)$  there corresponds a maximal finite-dimensional invariant subspace. Moreover, the maximal invariant subspace of the pair of operators  $(L, Q)$  corresponding to  $\lambda_k$  coincides with the maximal invariant subspace of the operator  $S$ , corresponding to the eigenvalue  $\lambda'_k = 1/\lambda_k$ .

Indeed, let  $\mathfrak{N}_k$  be the maximal invariant subspace of the operator  $S$ , corresponding to the eigenvalue  $\lambda'_k$ . As is known, in  $\mathfrak{N}_k$  one can choose a basis in which the transformation is written as a Jordan matrix. This basis consists of several vectors

$$f_{11}, f_{21}, \dots, f_{s_1 1}; f_{12}, f_{22}, \dots, f_{s_2 2}; \dots; f_{1q}, f_{2q}, \dots, f_{s_q q}. \quad (2)$$

Each chain forms a basis in one of the invariant subspaces into which the given maximal invariant subspace  $\mathfrak{N}_k$  decomposes. For the elements of each chain the equalities hold

$$Sf_{1\tau} = \lambda'_k f_{1\tau}, \quad Sf_{2\tau} = \lambda'_k f_{2\tau} + f_{1\tau}, \dots, \quad Sf_{s_\tau \tau} = \lambda'_k f_{s_\tau \tau} + f_{s_\tau - 1, \tau}.$$

If we recall that  $S = L^{-1}Q$ , we see that all elements of the basis (2) belong to  $D$ ; consequently,  $\mathfrak{N}_k \subset D$ . Since  $S\mathfrak{N}_k \subset \mathfrak{N}_k$ , we have  $Q\mathfrak{N}_k \subset L\mathfrak{N}_k$ , i.e.,  $\mathfrak{N}_k$  is an invariant subspace of the pair of operators  $(L, Q)$ . And since the eigenspace of the pair of operators  $(L, Q)$  corresponding to the eigenvalue  $\lambda_k$  coincides with the eigenspace of the operator  $S$  corresponding to the eigenvalue  $\lambda'_k = 1/\lambda_k$ , and any invariant subspace of the pair of operators  $(L, Q)$  is an invariant subspace of the operator  $S$ , the maximal invariant subspace of the pair of operators  $(L, Q)$ , corresponding to  $\lambda_k$ , coincides with  $\mathfrak{N}_k$ .

Thus, the question of the completeness of the linear span of the finite-dimensional invariant subspaces of the pair of operators  $(L, Q)$  reduces to the question of the completeness of the linear span of the invariant subspaces of the completely continuous operator  $S$ , or, equivalently, to the question of the completeness of the system of eigen- and associated elements of the operator  $S$ .

The latter, by arguments close to those of V. B. Lidskii <sup>(1)</sup>, under certain restrictions on the functions  $p(x)$  and  $q(x)$ , can be reduced to the question of the completeness of the system of eigen- and associated elements of the equation

$$g = Pg + \lambda Hg,$$

where  $H = A^{-1/2}QA^{-1/2}$  is a complete self-adjoint operator with finite absolute norm,  $P = A^{-1/2}p_2A^{-1/2}$  is a completely continuous operator; here  $A^{-1/2}$  is the square root of a self-adjoint positive operator—

operator  $A^{-1}$ , inverse to the operator  $A = -d^2/dx^2 + p_1(x)$ . By the theorem of M. V. Keldysh <sup>(3)</sup>, the system of eigen- and associated elements of such an equation is complete.

Thus, the following holds.

**Theorem.** Let the function  $p_1(x)$ , for some  $\alpha > 0$ , satisfy the condition

$$\lim_{|x| \rightarrow \infty} \frac{p_1(x)}{|x|^\alpha} \geq C > 0$$

and let the function  $q(x)$  be real, essentially bounded, and vanish only on a set of measure zero.

Then, for completeness in  $L^2(-\infty, +\infty)$  of the system of finite-dimensional invariant subspaces of the pair of operators  $(L, Q)$ , it is sufficient that

$$\lim_{|x| \rightarrow \infty} \frac{|p_2(x)|}{p_1(x)} = 0 \quad (3)$$

and, for all  $x$ ,

$$\frac{|p_2(x)|}{p_1(x)} < 1.$$

**Remark.** Completeness of the system of finite-dimensional invariant subspaces of the pair of operators  $(L, Q)$  also holds in the case where (3) is replaced by the condition

$$\lim_{|x| \rightarrow \infty} \frac{|p_2(x)|}{p_1(x)} = C,$$

where  $C < \sin \psi_0$ ,  $\psi_0 = \min\left(\frac{\pi}{2}, \frac{\alpha\pi}{\alpha+2}\right)$ .

Indeed, in this case the following theorem of J. E. Allakhverdiev <sup>(4)</sup> is applicable.

Let  $H$  be a complete, completely continuous normal operator of order  $\rho$ , whose eigenvalues are situated on a finite number of rays (let their number be  $k$ , and let the arguments of the rays be denoted by  $\varphi_1, \varphi_2, \dots, \varphi_k$  in increasing order); let  $A$  be a bounded operator with strictly bounded part  $B_0$ ,  $\|B_0\| < 1$ ; denote by  $\varepsilon_0$  the smallest angle for which  $\sin \varepsilon = \|B_0\|$ . If

$$2\varepsilon_0 < \min\left(|\varphi_{i+1} - \varphi_i|, \frac{\pi}{\rho}\right) \quad (i = 1, \dots, k) \quad (\varphi_{k+1} = \varphi_1),$$

then the system of eigen- and associated elements of the equation

$$y = (A + \lambda H)y$$

is complete in the space  $\mathcal{H}$ .

For any  $\varepsilon > \varepsilon_0$ , outside the angles  $\varphi_i - \varepsilon \leq \arg \lambda \leq \varphi_i + \varepsilon$  there can be only a finite number of eigenvalues of the operator  $A + \lambda H$ .

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## REFERENCES

1. V. B. Lidskii, *Tr. Mosk. matem. obshch.*, **9**, 45 (1960).
2. V. B. Lidskii, *Tr. Mosk. matem. obshch.*, **8**, 83 (1959).
3. M. V. Keldysh, *DAN*, **77**, No. 1 (1951).
4. J. E. Allakhverdiev, *Dokl. AN AzSSR*, **18**, No. 7 (1962).
5. L. S. Dashnits, *Zap. Matem. otd. fiz.-matem. fak. Kharkovsk. gos. univ. i Kharkovsk. matem. obshch.*, **25**, 139 (1957).

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