THE TRANSPORTATION PROBLEM IN TIME IN GRAPH THEORY

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Abstract

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THE TRANSPORTATION PROBLEM IN TIME IN GRAPH THEORY

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Let us call a finite connected graph $G = (X, U)$ without loops or multiple arcs a **transportation network** $G_{\|} = (x_0, X_n, z; U)$, if two vertices are distinguished in it: x_0 ($\Gamma^{-1}x_0 = \emptyset$), the entrance of the network, and z ($\Gamma z = \emptyset$), the exit, and with each arc $u \in U$ there are associated two integers: $l(u) \geq 0$, the length of the arc, and $c(u) \geq 0$, the capacity of the arc.

The transportation problem in time consists in finding the minimum time required to transport a load P from x_0 to z, the transportation being carried out along the arcs of the network.

A function $\varphi(u)$, defined on the set U and taking integer values, will be called a **flow** in the transportation network if:

1) $0 \leq \varphi(u) \leq c(u),\ u \in U$;
2)
$$\sum_{u\in U^-_{x_i}} \varphi(u) - \sum_{u\in U^+_{x_i}} \varphi(u) = 0 \qquad (x_i \in X_n,\ x_i \neq x_0,\ x_i \neq z).$$

The quantity

$$\Phi = \sum_{u\in U^-_z} \varphi(u) = \sum_{u\in U^+_{x_0}} \varphi(u)$$

is called the **value of the flow** φ. For a given transportation network the greatest value of the flow is equal to the least capacity of a cut (the Ford-Fulkerson theorem (^{1,5})), i.e.,

$$\max_{\varphi} \Phi = \min_{\substack{x_0\in A\\ z\notin A}} c(U^-_{\overline{A}}). \tag{1}$$

In the general case the flow in a transportation network is not defined uniquely (^{2}). Uniqueness can be achieved by imposing an additional condition on the flow $\varphi(u)$:

$$\sum \varphi(u) \cdot l(u) \leq \sum \varphi'(u) \cdot l(u), \tag{2}$$

where the summation is over all arcs entering a φ-cycle (2), and $\varphi'(u)$ is a flow of the same value as the flow $\varphi(u)$. The flow $\varphi(u)$ must satisfy condition (2) on all φ-cycles.

Theorem 1. *The minimum time for transporting a load P over a two-terminal transportation network is determined by the formula*

$$T = \left[\frac{P - 1 + \sum_{u \in U} \varphi(u) \cdot l(u)}{\Phi} \right], \tag{3}$$

where the square brackets denote taking the integer part.

For the proof of Theorem 1 let us consider the so-called parallel transportation networks, i.e., networks composed of parallel branches μ (Fig. 1). Denote such a network by $\widetilde{G}$. For this network a transportation plan ensures minimal transportation time if it satisfies the following condition.

Theorem 2. The time required to transport cargo P through the network G is minimal if and only if

$$\max_{i,j} \left| T_i - T_j \right| = 0 \text{ or } 1, \tag{4}$$

where T_i is the duration of transportation along the branch μ_i, $i, j = 1, 2, \dots, m$, where m is the number of branches. Transportation along all branches begins at the moment $t = 0$, and the size of the cargo portions, except possibly the last ones, is maximal.

From Theorem 2 there follows directly the solution of the transportation problem in time for parallel networks.

Theorem 3. The minimal time T for transporting cargo P through the network G is determined by the formula

$$T = \left[\frac{P - 1 + \sum_{i=1}^{m} \varphi(\mu_i) \cdot l_i}{\Phi} \right], \tag{5}$$

Fig. 1

where $\varphi(\mu_i)$ is the flow along the branch μ_i, and

$$l_i = \sum_{u \in \mu_i} l(u).$$

To solve the transportation problem in time for arbitrary networks, we introduce the concept of equivalence of networks. Networks G_1 and G_2 are called **equivalent** if 1) the flow through the network G_1 is equal in magnitude to the flow through G_2, and 2) equal quantities of cargo are transported through the networks G_1 and G_2 in equal time.

Theorem 4. For any arbitrary transportation network there exists an equivalent parallel transportation network.

From Theorems 3 and 4, Theorem 1 follows directly.

The result obtained here—the formula (3)—admits generalizations both to the case of bipartite networks of a more complicated nature, for example n-partite transportation networks, and to the case of certain special kinds of multipartite transportation networks. It is necessary to note, however, that in the latter case the concept of flow differs from that accepted in graph theory.

Remark. The results obtained in the article are valid for

$$P > \sum_{u \in U} \varphi(u) \cdot l(u).$$

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Note: Figure translations are in progress. See original paper for figures.

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