FREE DECOMPOSITIONS IN THE INTERSECTION OF PRIMITIVE CLASSES OF ALGEBRAS

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

T. M. BARANOVICH

FREE DECOMPOSITIONS IN THE INTERSECTION OF PRIMITIVE CLASSES OF ALGEBRAS

(Presented by Academician P. S. Aleksandrov on 25 XII 1963)

It is known that in some primitive classes of universal algebras there is a good theory of free decompositions, i.e., the theorem on the freedom of subalgebras of free algebras is valid; the theorem describing subalgebras of the free product of algebras, and usually the theorem following from them on the existence of isomorphic extensions for any two free decompositions. Let two primitive classes of algebras be given, $K_1 = (\Omega_1, \Lambda_1)$ and $K_2 = (\Omega_2, \Lambda_2)$, in each of which there is a good theory of free decompositions. The question arises: what will be the theory of free decompositions in the class $K = (\Omega_1 \cup \Omega_2, \Lambda_1 \cup \Lambda_2)$, which it is natural to call the **intersection** of the classes K_1 and K_2?

In the present note a theory of free decompositions is constructed in the intersection of such classes $K_1 = (\Omega_1, \Lambda_1)$ and $K_2 = (\Omega_2, \Lambda_2)$, the systems of operations of which Ω_1 and Ω_2 either do not intersect, or intersect in a nullary operation O, and then the identities Λ_1 and Λ_2 contain all possible identities $00 \cdots 0\omega = 0$, $\omega \in \Omega_1$ and $\omega \in \Omega_2$, respectively.

For simplicity of exposition all theorems are formulated and proved for the intersection of **two** primitive classes, although by the same method they can be proved for the intersection of **any finite number** of classes.

Let L be some primitive class of universal algebras with a system Ω of $n(\omega)$-ary operations ω, $n(\omega) \geq 0$, and a system of identities Λ. We shall consider two cases: either all algebras of the class L possess a zero subalgebra, and then we shall call it a **class of algebras with zero**, or not all algebras of the class L possess a zero subalgebra. In the second case the empty set will be regarded as a subalgebra of any L-algebra and called the **zero subalgebra** O, just as the zero subalgebra in the first case.

A set G_0 will be called an **Ω-partial algebra** if in it the elements $a_1 \dots a_n\omega$, $\omega \in \Omega$, $n = n(\omega) \geq 0$, are defined for some ordered systems of elements $a_1, \dots, a_n \in G_0$ and some $\omega \in \Omega$.

A subset X of an Ω-partial algebra G_0 will be called **Ω-pure** if in it the element $a_1 \dots a_n\omega$, $\omega \in \Omega$, $n = n(\omega) \geq 0$, is not defined for any set of elements $a_1, \dots, a_n \in X$ and for any operations $\omega \in \Omega$.

The L**-closure** of an Ω-partial algebra G_0 will mean the L-algebra

$$G = \overline{G_0}^L,$$

defined in the class L by the set G_0 and by the relations holding in G_0 with respect to the operations Ω (see (1)).

If

$$G_0 = \bigcup_{i \in I} H_i, \quad H_i \in L, \quad i \in I, \quad H_i \cap H_j = 0 \quad \text{for } i \neq j,$$

then $\overline{G_0}^L$ is the L**-free product**

$$\prod_{i \in I}^{*L} H_i$$

of the L-algebras H_i, $i \in I$. The L-closure of an Ω-pure set X is the L**-free algebra** $F_L(X)$ with system of L-free generators X.

Let us list the conditions that in what follows we shall impose on the class L:

I,1. A subalgebra of an L-free algebra with one generator is L-free.

I,2. A subalgebra U of the L-free product of two L-free algebras $F_L(X)$ and $F_L(Y)$ is the L-free product of the subalgebra $U \cap F_L(X)$ and an L-free algebra whose set of L-free generators contains, in particular, every element of $U \cap Y$.

II. The operation of L-free product is exact (see (^{2})), and to each L-algebra G there is uniquely assigned a group of automorphisms $A_L(G)$ such that:

1. The subalgebra U of the L-free product

$$G = \prod_{i \in I}^{*}{}_L H_i, \quad H_i \in L, \quad i \in I,$$

decomposes into the L-free product of the nonzero subalgebras $U \cap H_i$, $i \in I$, of certain nonzero subalgebras $U \cap H_i\alpha$, $\alpha \in A_L(G)$, $i \in I$, and of a certain L-free algebra.

2. For each nonzero intersection of the form $U \cap H_i\alpha$, $\alpha \in A_L(G)$, $i \in I$, the subalgebra of the form $(U \cap H_i\alpha)\alpha'$, $\alpha' \in A_L(U)$, enters into the L-free decomposition for U considered in II,1.

III. The group of automorphisms $A_L(G)$ is such that:

1. If U is a subalgebra of the L-algebra G, then for every automorphism $\alpha \in A_L(U)$ there exists an automorphism $\bar{\alpha} \in A_L(G)$ such that α and $\bar{\alpha}$ coincide on U.

A subalgebra of the form $U\alpha$, $\alpha \in A_L(G)$, will be called L-conjugate to the subalgebra U in the L-algebra G.

2. If
$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots$$
is an increasing sequence of subalgebras of the L-algebra G and
$$G = \bigcup_{n=1}^{\infty} G_n,$$
then for every automorphism $\alpha \in A_L(G)$ there exist an n and an $\alpha' \in A_L(G)$ such that
$$x\alpha = x\alpha'$$
for all $x \in G_n$.
3. For every homomorphism $\varphi : G \to H$, $G, H \in L$, and for every $\alpha \in A_L(G)$ there exists an automorphism $\alpha' \in A_L(H)$ such that
$$\alpha\varphi = \varphi\alpha'.$$

Let now $K_1 = (\Omega_1, \Lambda_1)$ and $K_2 = (\Omega_2, \Lambda_2)$ be two primitive classes of algebras. We shall assume that the sets of operations Ω_1 and Ω_2 either do not intersect, or intersect in a nullary operation 0, and in this case the identities Λ_n, $n = 1, 2$, contain identities of the form

$$0 \ldots 0\omega = 0, \quad n(\omega) \geq 0, \quad \omega \in \Omega_n.$$

Let
$$K = (\Omega_1 \cup \Omega_2, \Lambda_1 \cup \Lambda_2).$$

We shall denote by $F_n(X)$ the K_n-free algebra, $n = 1, 2$, by $F(X)$ the K-free algebra, by
$$\prod_{i \in I}^{*}{}_n H_i$$
the K_n-free product of K_n-algebras H_i, $i \in I$, and by
$$\prod_{i \in I}^{*} H_i$$
the K-free product.

Lemma. Let G_0 be an $\Omega_1 \cup \Omega_2$-partial algebra such that

$$G_0 \subseteq \bar{G}_0^{K_n}, \quad n = 1, 2,$$

and the K_n-free product of the K_n-algebra $\bar{G}_0^{K_n}$ with arbitrary K_n-free algebras is localized (see $(^2)$). Then the K-algebra $\bar{G}_0^{K}$ is the union of an increasing sequence of $\Omega_1 \cup \Omega_2$-partial algebras

$$G^0 \subseteq G^1 \subseteq \dots \subseteq G^{2k+1} \subseteq G^{2k+2} \subseteq \cdots ,$$

where

$$G^0 = G_0 \cup 0, \quad G^{2k+1} = \overline{G^{2k}}^{K_1}, \quad G^{2k+2} = \overline{G^{2k+1}}^{K_2}, \quad k = 0, 1, \dots .$$

Moreover, an element $a_1 \dots a_n \omega$, $\omega \in \Omega_2$, $n = n(\omega) \geq 0$, is defined in G^1 only in the case when $a_1, \dots , a_n \in G^0$ and the element $a_1 \dots a_n \omega$ was defined in G_0, while G^{2k} $[G^{2k+1}]$, $k = 1, 2, \dots$, is such an Ω_1- $[\Omega_2]$-partial algebra that the element $a_1 \dots a_n \omega$, $\omega \in \Omega_1$ $[\omega \in \Omega_2]$, $n = n(\omega) \geq 0$, is defined in it if and only if $a_1, \dots , a_n \in G^{2k-1}$ $[G^{2k}]$, and

$$G^2 = \bar{G}_0^{K_2} \underset{2}{*} F_2(G^1 \setminus G^0),$$

$$G^{2k+1} = G^{2k-1} \underset{1}{*} F_1(G^{2k} \setminus G^{2k-1}),$$

$$G^{2k+2} = G^{2k} \underset{2}{*} F_2(G^{2k+1} \setminus G^{2k}), \quad k = 1, 2, \dots .$$

Theorem 1. Let the classes K_1 and K_2 satisfy conditions I,1 and I,2. Then a K-subalgebra U of a K-free algebra $F(X)$ is K-free, and the set of its K-free generators contains the set $U \cap X$.

Theorem 2. Let conditions I,1; II,1 and III,1 hold for the classes K_1 and K_2. If

$$G = \prod_{i \in I}^{*} H_i$$

is the K-free product of K-algebras H_i, $i \in I$, then a K-subalgebra $U \subseteq G$ is representable in the form of a K-free product of nonunit intersections $U \cap H_i$, $i \in I$, of K-closures of some of its Ω_{n_1}-pure K_{n_2}-algebras $U \cap H_i \alpha$, $\alpha \in A_{n_2}(G)$, $n_1, n_2 = 1, 2$, $n_1 \neq n_2$, and of a certain K-free algebra.

Theorem 3. Let conditions I,1; II,1; II,2; III,1; III,2 hold for the classes K_1 and K_2. Then, in the K-free decomposition of the subalgebra U of the K-algebra

$$G = \prod_{i \in I}^{*} H_i$$

obtained in Theorem 2, for every nonzero intersection $U \cap H_i \alpha$, $\alpha \in A_n(G)$, there occurs the K-closure of a K_n-algebra of the form $(U \cap H_i \alpha)\alpha'$, $\alpha' \in A_n(U)$, $n = 1, 2$.

Theorem 4. Let conditions I,1; II,1—2; III,1—3 hold for classes with zeros K_1 and K_2. Then for any two K-free decompositions of a K-algebra G one can construct such continuations whose factors correspond bijectively to one another, and the corresponding factors either coincide, or are K-closures of mutually K_n-conjugate K_n-algebras, $n = 1, 2$, or are isomorphic K-free algebras.

It is known that conditions I,1—2; II,1—2; III,1—3 are satisfied in the following primitive classes: in the class of groups, if as $A_L(G)$ one takes the group of inner automorphisms of the given group G (^{3}), and also in the classes of nonassociative algebras (^{4,5}), loops (^{6}), multioperator algebras (^{7}), if as $A_L(G)$ for an algebra G of any of the listed classes one takes the group consisting of a single identity automorphism. It is easy to show that conditions I,1—2; II,1—2; III,1—3 are also satisfied in the class of algebras $L_0 = (\Omega, \Lambda)$, whose system of identities is either empty, or whose system of operations Ω contains the single nullary operation O, while the system of identities Λ consists of all possible identities of the form $00 \ldots 0\omega = 0$, $\omega \in \Omega$, and as $A_L(G)$ one must again take the identity automorphism. Moreover, conditions I,1—2 are satisfied in the class of abelian groups.

Let us now take as the class K_1 the class of groups, written additively, and as K_2 the class of algebras L_0 with zero. Theorems 1—4, as applied to these classes, turn into the corresponding theorems of A. G. Kurosh for multioperator groups (see Theorems 4, 5, 6 (^{8})). If as K_1 one takes the class of abelian groups, and as K_2 the same class of algebras L_0 with zero, then Theorem 1 turns into Theorem 4′ (^{8}). In general, by combining the operations and identities of a finite number of the above-listed classes of algebras satisfying conditions I,1—2; II,1—2; III,1—3, we obtain a class of algebras in which Theorems 1, 2, 3, 4 are valid. In particular, we obtain the theory of free decompositions of Ω-loops, which was considered by Higgins (^{9}).

Received
17 XII 1963

CITED LITERATURE

[1] A. I. Maltsev, *Izv. AN SSSR, ser. matem.*, **21**, 171 (1957).
[2] O. N. Golovin, *Tr. Mosk. matem. obshch.*, **12**, 413 (1963).
[3] A. G. Kurosh, *Theory of Groups*, 2nd ed., Moscow, 1953.
[4] A. G. Kurosh, *Matem. sborn.*, **20**, No. 2, 239 (1947).
[5] A. G. Kurosh, *Matem. sborn.*, **37**, No. 2, 251 (1955).
[6] G. Evans, *Amer. J. Math.*, **69**, No. 3, 499 (1947).
[7] A. G. Kurosh, *Sibirsk. matem. zhurn.*, **1**, No. 1, 62 (1960).
[8] A. G. Kurosh, *Acta Sci. Math. (Szeged)*, **21**, No. 3—4, 187 (1960).
[9] P. J. Higgins, *Proc. London Math. Soc.*, **6**, No. 23, 366 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the *original.*