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Abstract

Full Text

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NORMAL FORM OF DIFFERENTIAL EQUATIONS

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Let $\varphi_1(X), \ldots, \varphi_n(X)$ be power series in $x_1, \ldots, x_n$ without constant terms, converging in some neighborhood of the point $X = 0$. Then $X = 0$ will be a singular point of the system of ordinary differential equations

$$dx_i/dt = \varphi_i(X), \qquad i = 1, \ldots, n. \tag{1}$$

Let, further, $\xi_i(Y)$ and $\psi_i(Y)$ $(i = 1, \ldots, n)$ be power series in $y_1, \ldots, y_n$ without constant terms, converging in some neighborhood of the point $Y = 0$. Then the transformation

$$x_i = \xi_i(Y), \qquad i = 1, \ldots, n, \tag{2}$$

fixed at zero, transforms system (1) into the system

$$dy_i/dt = \psi_i(Y), \qquad i = 1, \ldots, n, \tag{3}$$

if the equalities

$$\sum_{l=1}^{n} \frac{\partial \xi_i}{\partial y_l} \psi_l(Y) = \varphi_i(\xi_1(Y), \ldots, \xi_n(Y)), \qquad i = 1, \ldots, n. \tag{4}$$

are satisfied.

In his dissertation (^{1}), Poincaré in fact proved that there exists a biholomorphic transformation (2) at zero which reduces system (1) to the form

$$dy_i/dt = \lambda_i y_i, \qquad i = 1, \ldots, n, \tag{5}$$

if the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\left\|\partial\varphi_i/\partial x_j\right\|_0$ satisfy the following three conditions: a) the normal Jordan form of the matrix $\left\|\partial\varphi_i/\partial x_j\right\|_0$ is diagonal; b) $\lambda_i \neq \sum_1^n \lambda_k p_k$ $(i = 1, \ldots, n)$ for any integers $p_1, \ldots, p_n \geq 0$, $\sum p_k > 1$; c) suppose the numbers $\lambda_1, \ldots, \lambda_n$ are represented by points in the complex plane; there exists in this plane a straight line passing through zero such that all the λ' s lie on one side of it.

The proof consists of two stages. First it is proved that there exist formal power series $\xi_i(Y)$ which formally satisfy the equalities (4), where $\psi_i = \lambda_i y_i$, i.e., if operations with formal power series are performed as with convergent ones, then identical formal series will stand on both sides of the equalities (4). The second stage consists in proving the convergence of the series ξ_i.

Generalizing Poincaré' s results, Picard (^{2}) showed that an invertible formal transformation of system (1) to the form (5) is possible if only conditions a) and b) are fulfilled. In this case system (5) is called the normal form of system (1). If, however, only condition c) is fulfilled, then, as Dulac (^{3}) showed, the λ' s can be numbered so that, by an invertible formal transformation (2), system (1) is reduced to the normal form

$$dy_i/dt = \lambda_i y_i + \sum a_{ip_1\cdots p_{i-1}} y_1^{p_1} \cdots y_{i-1}^{p_{i-1}}, \qquad i = 1, \dots, n,$$

where in the i-th equation the sum is taken over all integers $p_1, \dots, p_{i-1} \geq 0$ such that $\lambda_i = \lambda_1 p_1 + \dots + \lambda_{i-1} p_{i-1}$. Finally, Sternberg $^{(4)}$ proved that if condition a) is fulfilled, then there exists a formal change of variables (2) reducing the point transformation $x_i^* = \varphi_i(X)$, $i = 1, \dots, n$, to the form

$$y_i^* = \lambda_i y_i + \sum a_{ip_1\cdots p_n} y_1^{p_1} \cdots y_n^{p_n}, \qquad i = 1, \dots, n,$$

where in the i-th equality the sum is taken over all integers $p_1, \dots, p_n \geq 0$ such that $\sum p_k > 1$ and $\lambda_i = \lambda_1^{p_1} \cdots \lambda_n^{p_n}$. In the present paper an analogous form is established for an arbitrary system (1). Since questions of convergence of the series ξ will not be considered here, the convergence of the series φ and ψ is also immaterial. Therefore formal systems (1), (3) and formal transformations (2), which are a generalization of holomorphic systems and transformations at zero, will be considered.

In my paper $^{(5)}$ the following notation for systems of differential equations was introduced:

$$dy_i/dt = y_i g_i(Y) = y_i \sum_{Q \in N_i} g_{iQ} Y^Q, \qquad i = 1, \dots, n, \tag{6}$$

where $Q = (q_1, \dots, q_n)$ and $Y^Q = y_1^{q_1} \cdots y_n^{q_n}$. In the case considered here $y_i g_i(Y)$ are power series without constant terms; hence

$$N_i = \{Q: \text{ integers } q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n \geq 0,\ q_i \geq -1,\ \sum_1^n q_k \geq 0\},$$

$i = 1, \dots, n$. Denote

$$N = N_1 \cup \dots \cup N_n.$$

Theorem. There exists an invertible transformation (2) of system (1) into such a system (3) that: a) the matrix $\|\partial\psi_i/\partial y_j|_0\|$ is a normal Jordan form; b) when system (3) is written in the form (6), g_{iQ} are different from 0 only for those Q for which the scalar product $(Q, \Lambda) = 0$. Here $\Lambda = (\lambda_1, \dots, \lambda_n)$ is the vector of the diagonal elements of the matrix $\|\partial\psi_i/\partial y_j|_0\|$; φ, ψ, ξ are formal power series without constant terms.

The notation (6) and the theorem admit a simple geometric interpretation. To each coefficient $g_{iQ} \neq 0$ there corresponds a point Q in the n-dimensional real affine space R^n with coordinates $q_1, \dots, q_n$. Denote the set of all such points by $D(g_1, \dots, g_n)$. For an arbitrary system (6), $D(g_1, \dots, g_n) \subset N$. For the normal form, according to the theorem, $D(g_1, \dots, g_n) \subset N \cap L$, where L is the linear subspace in R^n orthogonal to the vectors $\operatorname{Re}\Lambda$ and $\operatorname{Im}\Lambda$, and the linear hull spanned by L has dimension $d < n$. Therefore, as proved in theorem 2, § 2 of my paper $^{(6)}$, there exists a birational transformation

$$z_i = y_1^{\alpha_{i1}} \cdots y_n^{\alpha_{in}} \quad (i = 1, \dots, n)$$

(i.e. α_{ij} are integers, $\det\|\alpha_{ij}\| = \pm 1$), transforming the normal form into the system

$$dz_i/dt = z_i g_i'(z_1, \dots, z_d), \qquad i = 1, \dots, n,$$

the first d equations of which form a system of order d, and $d\ln|z_{d+1}|/dt, \dots, d\ln|z_n|/dt$ are expressed through $z_1, \dots, z_d$.

Example 1. $n = 2$, $\lambda_1/\lambda_2 = -r/s$, where r and s are relatively prime natural numbers. Conditions a) and b) are not fulfilled. The equation $(Q, \Lambda) \equiv q_1\lambda_1 + q_2\lambda_2 = 0$ defines in the plane (q_1, q_2) a straight line which intersects N at the points $Q = (ks, kr)$, where $k \geq 0$ is an integer, since $N = \{Q :$ integers $q_1, q_2 \geq -1$ and $q_1 + q_2 \geq 0\}$. The normal form, according to the theorem, will be

$$dy_1/dt = y_1\left(\lambda_1 + \sum_{k=1}^{\infty} g_{1(ks,kr)} y_1^{ks} y_2^{kr}\right), \qquad dy_2/dt = y_2\left(\lambda_2 + \sum_{k=1}^{\infty} g_{2(ks,kr)} y_1^{ks} y_2^{kr}\right).$$

Since r and s are relatively prime, there exist integers u and v such that $ru - sv = 1$; then the birational transformation $z_1 = y_1^s y_2^r$, $z_2 = y_1^v y_2^u$ brings the normal form to the form

$$dz_1/dt = z_1 \sum_{k=1}^{\infty} \left(s g_{1(ks,kr)} + r g_{2(ks,ks)}\right) z_1^k,$$

$$dz_2/dt = z_2 \left[v\lambda_1 + u\lambda_2 + \sum_{k=1}^{\infty} \left(v g_{1(ks,kr)} + u g_{2(ks,kr)}\right) z_1^k\right].$$

Example 2. Picard' s result follows from the theorem, since condition b) means that the equation $(Q, \Lambda) = 0$ for $Q \in N$, $\sum q_i > 0$ has no solutions. Dulac' s result also follows from the theorem.

Lemma. There exists a formal transformation

$$x_i = y_i(1 + h_i(Y)), \qquad h_i(Y) = \sum h_{iQ} Y^Q, \qquad i = 1, \dots, n, \tag{7}$$

of the formal system

$$dx_i/dt = \lambda_i x_i + a_i x_{i-1} + x_i f_i(X), \qquad f_i(X) = \sum f_{iQ} X^Q, \qquad i = 1, \dots, n, \tag{8}$$

into such a formal system

$$dy_i/dt = \lambda_i y_i + a_i y_{i-1} + y_i g_i(Y), \qquad g_i(Y) = \sum g_{iQ} Y^Q, \qquad i = 1, \dots, n, \tag{9}$$

that $g_{iQ} = 0$ if $(Q, \Lambda) \neq 0$, and h_{iQ} for $(Q, \Lambda) = 0$ may be prescribed arbitrarily; then the remaining h_{iQ} and g_{iQ} are determined uniquely. Here $y_i h_i$, $x_i f_i$, and $y_i g_i$ are power series containing no terms below the second degree, and $\Lambda = (\lambda_1, \dots, \lambda_n)$.

Proof. By definition (4), the transformation (7) carries (8) into (9) if the formal equalities of series in $y_1, \dots, y_n$ are satisfied:

$$\sum_{l=1}^{n} \frac{\partial[y_i(1 + h_i(Y))]}{\partial y_l}(\lambda_l y_l + a_l y_{l-1} + y_l g_l) = \lambda_i y_i(1 + h_i) + a_i y_{i-1}(1 + h_{i-1}) +$$

$$+ y_i(1 + h_i) f_i(y_1(1 + h_1), \dots, y_n(1 + h_n)), \qquad i = 1, \dots, n.$$

Expanding the brackets, collecting like terms, and transferring some terms from the left-hand side to the right, we obtain

$$y_i g_i + y_i \sum_{l=1}^{n} \frac{\partial h_i}{\partial y_l} \lambda_l y_l = -h_i a_i y_{i-1} - h_i y_i g_i -$$

$$- y_i \sum_{l=1}^{n} \frac{\partial h_i}{\partial y_l} a_l y_{l-1} - y_i \sum_{l=1}^{n} \frac{\partial h_i}{\partial y_l} y_l g_l + a_i y_{i-1} h_{i-1} +$$

$$+ y_i(1 + h_i) f_i(y_1(1 + h_1), \dots, y_n(1 + h_n)), \qquad i = 1, \dots, n. \tag{10}$$

Writing out the coefficients of y_{iY}^{Q} in the i-th equality (10), we obtain

$$g_{iQ} + h_{iQ}(Q,\Lambda) = -h_{i,Q-E_{i-1}+E_i} a_i - \sum_{P+R=Q} h_{iP} g_{iR}$$

$$-\sum_{l=1}^{n} h_{i,Q-E_{l-1}+E_l}(q_l+1)a_l - \sum_{l=1}^{n} \sum_{P+R=Q} h_{iP} p_l g_{lR} +$$

$$+a_i h_{i-1,Q-E_{i-1}+E_i} + \{(1+h_i)f_i\}_Q, \qquad Q \in N_i, \qquad i = 1, \dots, n. \tag{11}$$

Here E_k denotes the k-th unit vector, and $\{(1+h_i)f_i\}_Q$ the coefficient of y_{iY}^{Q} in the series $y_i(1+h_i)f_i(y_1(1+h_1), \dots, y_n(1+h_n))$. The system of equalities (11) is equivalent to the system (10), since, as Q runs through N_i, the product y_{iY}^{Q} runs through all products of nonnegative powers of $y_1, \dots, y_n$.

The set of n-dimensional positive integral vectors is completely ordered by the relation: a vector P precedes a vector Q if the first positive nonzero one among the successive differences $\sum_1^n q_i - \sum_1^n p_i$, $q_1 - p_1, \dots,$ $q_{n-1} - p_{n-1}$ is positive. It is obvious that each $Q \in N$ is preceded by only a finite number of vectors from N. It is easy to see that only those h_{jP} and g_{iR} $(j = 1, \dots, n)$ for which the vectors P and R precede the vector Q enter the right-hand side of (11). This is true for the 1st, 3rd, and 5th terms on the right-hand side of (11), since $Q - E_{i-1} + E_l$ precedes the vector Q, and for the 2nd and 4th terms, since there the indices include only such P and R that $\sum p_i + \sum r_i = \sum q_i$ and $\sum p_i, \sum r_i > 0$; hence $\sum p_i, \sum r_i < \sum q_i$. Finally, $\{(1+h_i)+f_i\}_Q$ contains only such h_{jP} that $\sum p_i < \sum q_i$, since the series $x_{if}i(X)$ contains no linear terms.

The equalities (11) are trivially satisfied if $\sum q_i = 0$, since the equalities (10) contain no linear terms. The equalities (11) for $\sum q_i > 0$ will be satisfied if one sets $g_{iQ} = 0$, $h_{iQ} = (Q,\Lambda)^{-1} c_{iQ}$ for $(Q,\Lambda) \neq 0$; $g_{iQ} = c_{iQ}$, h_{iQ} arbitrary for $(Q,\Lambda) = 0$; $Q \in N_i$, $i = 1, \dots, n$. Here c_{iQ} denotes the right-hand side of the equalities (11). Thus, in the order indicated above with respect to Q, g_{iQ} and h_{iQ} $(i = 1, \dots, n)$ are determined in accordance with the assertion of the lemma.

Proof of the theorem. The transformation whose existence is asserted by the theorem is the result of successively carrying out: a) the linear transformation that brings the matrix $\|d\varphi_i/dx_j\|_0$ to the normal Jordan form (7), and b) the transformation of the lemma, which is now possible, since the normal Jordan form is a special case of the matrix of the linear part of the system (8).

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Note: Figure translations are in progress. See original paper for figures.