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Abstract

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GEOPHYSICS

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A NUMERICAL ALGORITHM FOR SOLVING THE EQUATIONS OF WEATHER FORECASTING

In the work (1) a theoretical model of weather forecasting was formulated and a basic algorithm was developed for solving the system of equations of the dynamics of atmospheric processes on the basis of the method of splitting multidimensional operators into a sequence of one-dimensional ones. As a result, the problem of forecasting the fields of meteorological elements is reduced to the successive solution of the following problems:

$$\begin{aligned}
&\frac{1}{\Delta t}\left(u^{j+1/4}-u^{j}\right)+u^{j}u_{x}^{j+1/4}-\mu u_{xx}^{j+1/4}=0,\\
&\frac{1}{\Delta t}\left(v^{j+1/4}-v^{j}\right)+u^{j}v_{x}^{j+1/4}-\mu v_{xx}^{j+1/4}=0,\\
&\frac{1}{\Delta t}\left(T^{j+1/4}-T^{j}\right)+u^{j}T_{x}^{j+1/4}-\mu_{T}T_{xx}^{j+1/4}=0;
\end{aligned}\tag{1}$$

$$\begin{aligned}
&\frac{1}{\Delta t}\left(u^{j+2/4}-u^{j+1/4}\right)+v^{j}u_{y}^{j+2/4}-\mu u_{yy}^{j+2/4}=0,\\
&\frac{1}{\Delta t}\left(v^{j+2/4}-v^{j+1/4}\right)+v^{j}v_{y}^{j+2/4}-\mu v_{yy}^{j+2/4}=0,\\
&\frac{1}{\Delta t}\left(T^{j+2/4}-T^{j+1/4}\right)+v^{j}T_{y}^{j+2/4}-\mu_{T}T_{yy}^{j+2/4}=0;
\end{aligned}\tag{2}$$

$$\begin{aligned}
&\frac{1}{\Delta t}\left(u^{j+3/4}-u^{j+2/4}\right)+\tau^{j}u_{p}^{j+3/4}-\left(\lambda p^{2}u_{p}^{j+3/4}\right)_{p}=0,\\
&\frac{1}{\Delta t}\left(v^{j+3/4}-v^{j+2/4}\right)+\tau^{j}v_{p}^{j+3/4}-\left(\lambda p^{2}v_{p}^{j+3/4}\right)_{p}=0,\\
&\frac{1}{\Delta t}\left(T^{j+3/4}-T^{j+2/4}\right)-\left(\lambda_{T}p^{2}T_{p}^{j+3/4}\right)_{p}=0;
\end{aligned}\tag{3}$$

$$\begin{aligned}
&\frac{1}{\Delta t}\left(u^{j+1}-u^{j+3/4}\right)-lv^{j+1}=-H_x^{j+1},\\
&\frac{1}{\Delta t}\left(v^{j+1}-v^{j+3/4}\right)+lu^{j+1}=-H_y^{j+1},\\
&\frac{1}{\Delta t}\left(T^{j+1}-T^{j+3/4}\right)-\frac{\gamma_a-\gamma}{g}RT\frac{\tau^{j+1}}{p}=0,\\
&u_x^{j+1}+v_y^{j+1}+\tau_p^{j+1}=0,\\
&T^{j+1}=-\frac{p}{R}H^{j+1}.
\end{aligned} \tag{4}$$

Here, for simplicity, it is assumed that the nonadiabatic heat inputs are equal to zero.

Analysis of the systems of equations (1)–(4) shows that the first three systems satisfy neither the continuity equation nor the quasistatic relation. However, the solutions on the lines t_i, t_{i+1}, etc., do satisfy these relations. Functions with fractional indices have an auxiliary significance in solving the general problem and cannot, generally speaking, be of independent interest in interpreting the results.

The boundary conditions for the systems of equations (1)–(4) may be stated as follows. On the lateral surface, the values of the functions are prescribed

$u,\ v,\ T,\ H$ for all moments of time. At the Earth' s surface ($p=p_0$), $\tau=\tau_0$; at the upper boundary of the atmosphere, $\tau=0$. As initial data one takes $u(x,y,p,0),\ v(x,y,p,0),\ T(x,y,p,0)$.

We pass to a finite-difference formulation of the problem (in x,y). To this end, we write each equation of the system (1), (2) in the form

$$\varphi_k+r_k(\varphi_{k+1}-\varphi_{k-1})-\nu(\varphi_{k+1}-2\varphi_k+\varphi_{k-1})=f_k, \tag{5}$$

where

$$r_k=\frac{\sigma_k\psi_k^j}{2}\frac{\Delta t}{h},\qquad \nu=\mu\frac{\Delta t}{h^2},\qquad \sigma_k\psi_k^j=\frac{1}{2}\left(\psi_{k-1}^j+\psi_{k-1}^j\right),\qquad h=\Delta x+\Delta y.$$

We note that the operator σ_k plays a significant role in constructing an effective finite-difference system of equations. This is connected with the fact that the exact formula

$$\nabla_k(\varphi\psi)=\sigma_k\varphi\nabla_k\psi+\sigma_k\psi\nabla_k\varphi,\qquad \nabla_k\psi=\psi_{k+1}-\psi_{k-1}, \tag{6}$$

holds, which makes it possible to carry out various transformations of the system of difference equations while preserving their divergent structure. For unsplit equations in difference form the formulated assertion is strict; for the split system, preservation of the divergent structure of the difference equations turns out to be approximate, though with a good degree of accuracy.

We write the three-point difference equation (5) in the form

$$a_k\varphi_{k+1} + b_k\varphi_k - c_k\varphi_{k-1} = f_k, \tag{7}$$

where $a_k = r_k - \nu,\ b_k = 1 - 2\nu,\ c_k = r_k + \nu.$

The solution of the difference equation (7) with the boundary condition

$$\varphi_0 = A, \qquad \varphi_n = B \tag{8}$$

is sought by means of the factorization method

$$\beta_{k+1} = -\frac{a_k}{b_k + c_k\beta_k}, \qquad z_{k+1} = \frac{c_k z_k + f_k}{b_k + c_k\beta_k}, \qquad \varphi_k = \beta_{k+1}\varphi_{k+1} + z_{k+1}. \tag{9}$$

In the most unfavorable case for computation, when turbulent exchange is absent, formulas (9) prove effective for implementing the solution provided the Courant condition is satisfied,

$$\sigma_k\psi^j\frac{\Delta t}{h} \leqslant 1.$$

Condition (10), for weather-forecasting problems, is not a serious restriction.

When considering the system of equations (4), one has to deal with a nonuniform grid of points in the coordinate p. This is caused by the specific nature of atmospheric processes in the planetary layer. The construction of higher-accuracy difference schemes on nonuniform grids can be carried out by the algorithm formulated in [4]. As a result, we again arrive at three-point difference equations (7) with condition (8).

We now turn to the most difficult question—the solution of the system of equations (4) under the corresponding boundary conditions. As shown in [1], the system of equations (4) reduces to a single equation

$$\frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial H^{j+1}}{\partial p} + \frac{\alpha^2}{1+\alpha^2}\left(H^{j+1}_{xx} + H^{j+1}_{yy} + \alpha_x H^{j+1}_y - \alpha_y H^{j+1}_x\right) = -f^{j+1}, \tag{10}$$

where

$$f^{j+1} = \frac{\partial}{\partial p}\frac{pR}{m^2}T^{j+3/4} - \frac{\alpha^2}{1+\alpha^2}\left[(u+\alpha v)_x^{j+3/4} + (v-\alpha u)_y^{j+3/4}\right], \qquad (11)$$

under the condition

$$p\,\frac{\partial H^{j+1}}{\partial p} - \frac{\gamma_a - \gamma}{g}\,RH^{j+1} = C^j, \qquad p = p_0,$$
$$p\,\frac{\partial H^{j+1}}{\partial p} = 0, \qquad p = 0. \qquad (12)$$

Here H and T are, respectively, the deviation of the height of the isobaric surface and of the temperature from normal (climatic) values; C^j is a known function of the coordinates (x, y, t_j). The notation for the other quantities is as in [1].

To solve problem (10)–(12), we shall formulate the following economical relaxation method, based on the splitting method (see [1]).

Let us introduce a new variable ξ and transform equation (10) into the following:

$$\frac{\partial H}{\partial \xi} = \frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial H}{\partial p} + \frac{\alpha^2}{1+\alpha^2}\left(H_{xx} + H_{yy} + \alpha_x H_y - \alpha_y H_x\right) + f \qquad (13)$$

under the conditions

$$p\,\frac{\partial H}{\partial p} + \frac{\gamma_a - \gamma}{g}\,RH = C, \qquad p = p_0,$$
$$p\,\frac{\partial H}{\partial p} = 0, \qquad p = 0, \qquad (14)$$

where $H = H(x, y, p, \xi)$, $H^{j+1} = H(x, y, p, \infty)$.

Choose an arbitrary interval $\Delta\xi = \xi_{n+1} - \xi_n$, within which we carry out the splitting of problem (13), (14):

$$\frac{H_{n+1/3} - H_n}{\Delta\xi} = \frac{\alpha^2}{1+\alpha^2}\left(H_{xx} - \alpha_y H_x\right)_{n+1/3},$$
$$\frac{H_{n+2/3} - H_{n+1/3}}{\Delta\xi} = \frac{\alpha^2}{1+\alpha^2}\left(H_{yy} + \alpha_x H_y\right)_{n+2/3},$$
$$\frac{H_{n+1} - H_{n+2/3}}{\Delta\xi} = \frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial H_{n+1}}{\partial p} + f. \qquad (15)$$

Equations (15), together with the corresponding boundary conditions, are written in finite-difference form in the variables (x, y, p), similarly to what was indicated above. The three-point difference equations are solved by the factorization method.

The question of choosing the relaxation parameter $1/\Delta\xi$ deserves special attention. In view of the specifics of problem (13), (14), it is possible to formulate a rapidly convergent process.

We choose the relaxation parameter from the condition

$$\frac{1}{\Delta\xi} = \frac{1}{1+\alpha^2}\frac{4}{h_1^2}. \tag{16}$$

With this choice, the iterative process converges with practically optimal speed. If, furthermore, the extrapolated value H^{j+1} is taken as the initial approximation, i.e.,

$$H_0 = H^j + \Delta H^j,$$

where $\Delta H^j = H^j - H^{j-1}$, then the accuracy practically required for computations is already achieved after carrying out 1-2 iterations. Thus we arrive at one of the varieties of semi-iterative schemes.

After the functions H^{j-1} have been found, the values of the components of the vector

the velocities are determined by the formulas

$$u^{j+1} = \frac{1}{1+\alpha^2}(u+\alpha v)^{j+3/4} - \frac{\Delta t}{1+\alpha^2}(H_x + \alpha H_y)^{j+1},$$

$$v^{j+1} = \frac{1}{1+\alpha^2}(v-\alpha u)^{j+3/4} - \frac{\Delta t}{1+\alpha^2}(H_y - \alpha H_x)^{j+1}. \tag{17}$$

Thus, the problem of weather forecasting is represented as a sequence of elementary algorithms that are efficiently implementable on an electronic computer.

It should be noted that further complication of the mathematical formulation of the problem fully preserves the general algorithm formulated here, to which new elementary algorithms are merely added additively, reducing to the solution of the simplest problems of linear algebra.

In conclusion, we note that a theoretical justification of the formulated splitting method for the equations of the dynamics of atmospheric processes proved possible when the original nonlinear problem was replaced by the corresponding linear problem with constant coefficients.

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CITED LITERATURE

1. G. I. Marchuk, DAN, **155**, No. 5 (1964).

2. G. I. Marchuk, *Methods for Calculating Nuclear Reactors*, Moscow, 1961.

3. S. K. Godunov, V. S. Ryabenkii, *Introduction to the Theory of Difference Schemes*, Moscow, 1962.

4. A. N. Tikhonov, A. A. Samarskii, *Computational Mathematics and Mathematical Physics*, **2**, 5 (1962).

Note: Figure translations are in progress. See original paper for figures.