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Abstract

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PERIODIC AND LOCALLY NILPOTENT SUBGROUPS OF TOPOLOGICAL GROUPS

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Let G be an abstract periodic group. As usual, $\pi(G)$ denotes the set of all prime divisors of the orders of elements of the group G. In particular, if $g \in G$, then $\pi(g)$ is the set of prime divisors of the order of g. If π is some set of primes and $\pi(g) \subseteq \pi$, then g is called a π-element, and if $\pi(G) \subseteq \pi$, then G is called a π-group. Now let Γ be a topological group with identity e. An element $g \in \Gamma$ is called a topological m-element if the sequence $g^m, g^{m^2}, \ldots, g^{m^k}, \ldots \to e$ in the topology of the group Γ. If, however, $g^{m^\nu} = e$, then the element g is called a discrete m-element. It is clear that every discrete m-element is topological, and in any topology. A topological m-element is called a topological π-element if all prime divisors of m belong to π. A subgroup $\Pi \subset \Gamma$ is called a topological (discrete) π-subgroup if all its elements are topological (respectively discrete) π-elements. In what follows, p is a prime number, and π is an arbitrary set of prime numbers.

The following well-known problems arise naturally:

1. When are topological Sylow p-subgroups conjugate in a topological group G?

1′. When are discrete Sylow p-subgroups conjugate in G?

2. When are topological Sylow π-subgroups conjugate in a locally solvable topological group Γ?

2′. When are discrete Sylow π-subgroups conjugate in Γ?

D. Dantzig (3) showed that for zero-dimensional bicompact groups problem 1 has a positive solution. A somewhat more general result was obtained in (2). Recently, in (4), problem 2 was also solved for the case of bicompact zero-dimensional groups.

In the present paper, relying on the results of A. I. Mal' tsev (6,7), K. Iwasawa (8), A. Gleason (9,10), H. Yamabe (11,12), see also (13), concerning Hilbert' s fifth problem and the structure of locally bicompact groups, and on the ideas of the author' s papers (1,14), it is shown that problems 1 and 2 have positive solutions for a very broad class of topological groups, whereas problems 1′ and 2′ have negative solutions already in the simplest case. It is also shown that in solvable periodic and solvable algebraic linear groups the Sylow π-subgroups are conjugate. Of substantial importance is the general Cartan–Mal' tsev–Iwasawa theorem established in the paper. In conclusion, a certain class of solvable linear groups is constructed. With the help of groups of this class a number of counterexamples are obtained, confirming the definitive character of the results of the paper; an example is also constructed of a solvable linear group possessing an infinite number of pairwise non-isomorphic Sylow p-subgroups, which gives a negative answer to a question of D. A. Suprunenko. In the proofs, besides the results mentioned above, use is made of the theory of projective ([17]) and transfinite ([18]) limits of topological groups, invariant integration on abstract groups ([21]), and some results from the theory of linear groups.

The basic terminology of the paper agrees with the generally accepted one ([17,18,20]). All topological groups considered below are locally bicompact; G_0 is the connected component of the identity of the group G.

A topological group Γ is called **projectively solvable** if every neighborhood of the identity U contains such a normal divisor H_U that the factor group Γ/H_U is solvable. It is not hard to show that for groups Γ with bicompact factor group Γ/Γ_0, every locally solvable group, as well as every RN^*-group Γ, is projectively solvable.

Let us formulate the main results of the paper.

Theorem A. *Let G be a topological group with bicompact factor group G/G_0. Then the topological Sylow p-subgroups in G are conjugate.*

Theorem B. *Let Γ be a projectively solvable group with bicompact factor group Γ/Γ_0. Then the topological Sylow π-subgroups in Γ are conjugate.*

Let us immediately note some corollaries following from Theorems A and B.

Corollary 1. *In a bicompact topological group the topological Sylow p-subgroups are conjugate.*

Corollary 2. *In a connected topological group the Sylow p-subgroups are conjugate.*

Corollary 3. *In a locally solvable topological group Γ with bicompact factor group Γ/Γ_0, the topological Sylow π-subgroups are conjugate.*

In the theory of Lie groups, an important role is played by the Cartan—Mal' tsev—Iwasawa theorem ([6],[8]). It turns out that for locally bicompact groups the following general Cartan—Mal' tsev—Iwasawa theorem holds, following from results of ([6],[8],[12],[15]).

Theorem C. *Let G be a topological group with bicompact factor group G/G_0. Then every bicompact subgroup of the group G is contained in a maximal bicompact subgroup. All maximal bicompact subgroups are conjugate in G. If B is one of them, then*

$$G = B \cdot H_1(t) \cdots H_r(t),$$

where $H_i(t)$, $i = 1, \ldots, r$, *are one-dimensional vector groups, and the mapping*

$$B \times H_1(t) \times \cdots \times H_r(t) \to B \cdot H_1(t) \cdots H_r(t)$$

is a homeomorphism.

In what follows, Theorem C is of great importance. One can formulate a somewhat more general result; for this purpose we introduce the following definition.

A group G is called a **group of CMI type** if every bicompact subgroup of it is contained in a maximal bicompact subgroup, and these are conjugate in G.

Theorem C′. *A topological group* G *is a group of CMI type if and only if the factor group* G/G_0 *is a group of CMI type.*

As examples show (some of them will be constructed at the end of the paper), Theorems A, B, C do not admit strengthening and are already false for arbitrary bicompactly generated groups.

The proofs of Theorems A and B rely on Theorem C and are carried out according to the following general reduction scheme: arbitrary topological groups → Lie groups → linear Lie groups → compact Lie groups. This scheme, in combination with Theorem C, appears to be very effective in many structural questions of the theory of locally bicompact groups.

We give the most important results on which the proofs of Theorems A and B are based. Most of them are also of independent interest.

Theorem 1. *A locally nilpotent group of semisimple automorphisms of an arbitrary finite-dimensional Lie algebra over a field of characteristic zero leaves invariant some Cartan subalgebra.*

Theorem 1 generalizes Theorem 2 from ([1]) and supplements the results of A. Borel and G. Mostow ([16]).

Corollary. *A locally nilpotent subgroup* H *of a compact Lie group* Γ *belongs to the normalizer* $N_\Gamma(T)$ *of some maximal torus* T.

Since every torus is a complete group, it follows from Lemma 1 in ([1]) and the local nilpotence of p-subgroups of compact Lie groups that

Theorem 2. In every compact Lie group the Sylow p-subgroups are conjugate.

The following theorem shows that in Lie groups there is no point in distinguishing between topological and discrete π-subgroups.

Theorem 3. In a Lie group, a topological π-element is a discrete π-element.

It is known that topological Sylow p-subgroups of a zero-dimensional bicompact group are closed. It is easy to see that for connected groups this is no longer true even in the case of abelian groups.

Theorem 4. Let Γ be a topological group with bicompact quotient group Γ/Γ_0. Then the closure $\overline{S}$ of every π-subgroup $S \subset \Gamma$ is a bicompact group.

From Theorem C and Theorem 4 it follows that the proof of Theorems A and B reduces to the case of a bicompact group.

At the same time, Theorems C, 2, and 4 already imply

Theorem 5. In every Lie group whose connected component has finite index, the Sylow p-subgroups are conjugate.

Simple examples show (see below) that the finiteness of the index in Theorem 5 is essential.

Let now B be an arbitrary bicompact group, and let S_1, S_2 be topological Sylow p-subgroups. The concept of the projective closure of the groups S_1, S_2, denoted respectively by $\widetilde{S}_1, \widetilde{S}_2$, is introduced in the natural way and is based on representing the group B as a projective limit of compact Lie groups.

The bicompact case is based on the following two theorems.

Theorem 6. The projective closures $\widetilde{S}_1$ and $\widetilde{S}_2$ are conjugate in B.

Theorem 7. Let S be an arbitrary p-subgroup in B. Then in $\widetilde{S}$ the topological Sylow p-subgroups are conjugate.

From Theorems 6 and 7 it follows obviously that

Theorem 8. In every bicompact group the topological Sylow p-subgroups are conjugate.

Theorem A now follows in an obvious way from Theorems C, 4, and 8.

We pass to the consideration of π-subgroups. First we solve the question of conjugacy of Sylow π-subgroups in solvable linear groups.

Theorem 9. In every solvable periodic linear group G, the Sylow π-subgroups are conjugate, and the Sylow π- and $\pi' = \pi(G) \setminus \pi$-subgroups are mutually complemented in G.

Theorem 9 generalizes the well-known theorem of P. Hall (^{5}) for finite groups and gives a positive answer to the question posed by D. A. Suprunenko at the Third All-Union Colloquium on General Algebra. It is interesting to note that the proof of Theorem 9 is carried out by the method applied by the author in (^{1}) in studying the structure of solvable algebraic groups.

From the results of (^{1}) and Theorem 9 it follows that

Theorem 10. In a solvable algebraic linear group the Sylow π-subgroups are conjugate.

Since in the full linear group $L_n(P)$ over an algebraically closed field the number of conjugacy classes of maximal solvable subgroups is finite (^{22}), Theorem 10 implies

Theorem 11. In $L_n(P)$ the number of conjugacy classes of solvable Sylow π-subgroups is finite.

If $2 \notin \pi$, then every π-subgroup of $L_n(P)$ is solvable, as follows from Thompson and Feit's positive solution of the Burnside problem and from the local finiteness of a periodic linear group (^{23}). We also note that the algebraicity condition in Theorem 10 is essential.

From (^{1}) and Theorem 9 it follows that

Theorem 12. In a solvable compact Lie group the Sylow π-subgroups are conjugate.

Analogously to Theorems 5 and 8, relying on Theorem 12, the following assertions are obtained.

Theorem 13. *In a solvable Lie group whose connected component has finite index, the Sylow π-subgroups are conjugate.*

Theorem 14. *In a bicompact projectively solvable group, the topological Sylow π-subgroups are conjugate.*

The proof of Theorem B is obtained by applying Theorems C, 4 and 14. In conclusion we shall construct one class of two-step solvable linear groups. Let ε be a primitive root of degree p^α of 1, and let $f(\varepsilon)$ be an arbitrary integral polynomial in ε. Denote by S_{p^α} the group generated by the matrices

$$a = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, \qquad b_f = \begin{bmatrix} 1 & f(\varepsilon) \\ 0 & 1 \end{bmatrix}$$

for all possible f, and by S_{p^∞} the union of the increasing sequence of groups

$$S_p \subset S_{p^2} \subset \cdots \subset S_{p^\alpha} \subset \cdots.$$

Theorem 15. *In the group S_{p^α} there are exactly p^α classes of conjugate Sylow p-subgroups, and α classes of isomorphic Sylow p-subgroups. In S_{p^∞} there is an infinite number of pairwise nonisomorphic Sylow p-subgroups.*

It is not difficult to see that S_{p^α} is a two-step solvable bicompactly generated Lie group, and for S_{p^α} Theorems A, B, C are not true. The groups S_{p^α} also make it possible to solve Problems $1'$ and $2'$ negatively already in the simplest case of two-step solvable bicompact zero-dimensional groups. A topology defined by a system of normal divisors can be introduced naturally into the group S_{p^α}. As is known (see (19), Theorem 11), the group S_{p^α} is then completable in the sense of A. Weil. Let $\hat{S}_{p^\alpha}$ be the Weil completion of the group S_{p^α}.

Theorem 16. *$\hat{S}_{p^\alpha}$ is a two-step solvable bicompact zero-dimensional group. In $\hat{S}_{p^\alpha}$, the discrete Sylow p-subgroups split into p^α conjugacy classes and α isomorphism classes.*

Although in the general case the discrete π-subgroups are not conjugate, nevertheless the following is true.

Theorem 17. *In a topological solvable group whose connected component has finite index, the Sylow π-subgroups, both discrete and topological, are conjugate.*

The conjugacy of the topological π-subgroups follows from Theorem B. The proof of the conjugacy of the discrete Sylow π-subgroups is carried out in an analogous way. Only one additional result on complete topological groups is used.

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Note: Figure translations are in progress. See original paper for figures.