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Abstract

Full Text

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MATHEMATICS

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On Some Properties of Linear Operators Acting in the Spaces \mathcal{L}_p

(Presented by Academician A. Yu. Ishlinskii on 12 VI 1964)

In recent years a number of new theorems have been established on the continuity and complete continuity of linear and, in particular, linear integral operators⁽¹⁻⁶⁾. In these theorems an essential role is played by the property of compactness of operators with respect to measure. The author has undertaken an attempt at a detailed analysis of this property. It has turned out that the systematic use of the property of compactness with respect to measure for linear operators, in combination with the concept of regularity (see⁽⁷⁾), makes it possible to strengthen a number of known theorems and to establish some new assertions. The article also analyzes certain other properties of linear operators closely connected with the property of compactness with respect to measure.

Let a measure μ be given on a set Ω . As usual, by \mathcal{L}_p ($0 < p < \infty$) we denote the class of measurable functions for which

$$\|x\|_p = \left\{ \int_{\Omega} |x(t)|^p d\mu(t) \right\}^{1/p} < \infty.$$

For $p \geq 1$ the class \mathcal{L}_p is a Banach space; for $p < 1$ the class \mathcal{L}_p is not a Banach space, but it is a complete metric space with metric $\rho(x, y) = \|x - y\|_p^p$. By \mathcal{L}_{∞} we denote the Banach space of essentially bounded functions with norm

$$\|x\|_{\infty} = \text{vrai sup}_{t \in \Omega} |x(t)|.$$

By P_D we denote the operator of multiplication by the characteristic function of a set $D \subset \Omega$; this operator is continuous in each of the spaces \mathcal{L}_p .

We consider linear continuous operators acting from the space $\mathcal{L}_p = \mathcal{L}_p(\Omega_1)$ into the space $\mathcal{L}_q = \mathcal{L}_q(\Omega_2)$. Here Ω_1 and Ω_2 are two sets with finite and non-atomic measures μ_1 and μ_2 (non-atomicity of a measure means that every measurable set can be divided into two parts of equal measure).

1. A linear continuous operator A , acting from \mathcal{L}_p into \mathcal{L}_q , will be called **improving** if

$$\lim_{\mu_2(F) \rightarrow 0} \|P_F A\| = 0. \quad (1)$$

Improving operators are operators that transform norm-bounded sets \mathfrak{M} of functions of the space \mathcal{L}_p into sets $A\mathfrak{M}$ of functions of the space \mathcal{L}_q with uniformly absolutely continuous norms. There are no nonzero improving operators acting from \mathcal{L}_p into \mathcal{L}_∞ .

A linear continuous operator A , acting from \mathcal{L}_p into \mathcal{L}_q , will be called **co-improving** if

$$\lim_{\mu_1(G) \rightarrow 0} \|AP_G\| = 0. \quad (2)$$

Co-improving operators are operators that transform every norm-bounded sequence of functions $x_n \in \mathcal{L}_p$ converging in measure to x_0 into a sequence of functions Ax_n , strongly converging

to Ax_0 in \mathcal{L}_q . There do not exist nonzero co-improving operators acting from \mathcal{L}_1 to \mathcal{L}_q for $q \geq 1$.

If $1 \leq p < \infty$, $1 \leq q \leq \infty$, then an operator A acting from \mathcal{L}_p to \mathcal{L}_q is improving if and only if the adjoint operator A^* , acting from $\mathcal{L}_{q'}$ to $\mathcal{L}_{p'}$, is co-improving.

We shall say that a continuous operator A acting from \mathcal{L}_p to \mathcal{L}_q has the **Ando property** if

$$\lim_{\mu_1(F) + \mu_2(G) \rightarrow 0} \|P_G A P_F\| = 0. \quad (3)$$

Improving and co-improving operators have the Ando property.

Theorem 1. *Let $1 \leq q < p < \infty$. Then every linear continuous operator A acting from \mathcal{L}_p to \mathcal{L}_q has the Ando property.*

This assertion for integral operators with nonnegative kernels was proved by T. Ando in [6].

2. A linear operator A acting from \mathcal{L}_p to \mathcal{L}_q is called **regular** (see [7]) if it can be represented in the form

$$A = A_1 - A_2,$$

where A_1 and A_2 are positive linear operators acting from \mathcal{L}_p to \mathcal{L}_q . Every regular operator A acting from \mathcal{L}_p to \mathcal{L}_q is continuous (see [8]). An integral

operator with kernel $K(t, s)$, acting from \mathcal{L}_p to \mathcal{L}_q , is regular if and only if the integral operator with kernel $|K(t, s)|$ acts from \mathcal{L}_p to \mathcal{L}_q .

Theorem 2. *Let a linear regular operator A act from \mathcal{L}_p to \mathcal{L}_q , where $0 < p \leq \infty$, $0 < q < \infty$, and have the Ando property. Then A is an improving operator.*

Theorem 3. *Let a linear regular operator A act from \mathcal{L}_p to \mathcal{L}_q , where $1 < p \leq \infty$, $0 < q \leq 1$. Then A is a co-improving operator.*

In particular, it follows from Theorems 1-3 that every regular operator A acting from \mathcal{L}_p to \mathcal{L}_q for $1 < p < \infty$, $p > q$, is simultaneously improving and co-improving.

Theorem 4. *Let A be a regular linear operator acting from \mathcal{L}_p to \mathcal{L}_q ($0 < p, q < \infty$).*

Then the operator A maps every set \mathfrak{M} of functions with equiabsolutely continuous norms in \mathcal{L}_p into the set $A\mathfrak{M}$ of functions with equiabsolutely continuous norms in \mathcal{L}_q .

This theorem was proved jointly with E. I. Pustynnik (see also [5]).

3. A linear operator A acting from \mathcal{L}_p to \mathcal{L}_q is called **compact in measure** (see [3]) if it maps every norm-bounded set $\mathfrak{M} \subset \mathcal{L}_p$ into a set compact in measure. For compactness in measure of an operator A acting from \mathcal{L}_p , where $p > 1$, to \mathcal{L}_q , it is sufficient that A transform every sequence $x_n \in \mathcal{L}_p$ weakly converging to x_0 into a sequence Ax_n converging to Ax_0 in measure. For $q \geq 1$ the converse assertion is true: every operator acting from \mathcal{L}_p to \mathcal{L}_q and compact in measure transforms a sequence $x_n \in \mathcal{L}_p$ weakly converging to x_0 into a sequence Ax_n converging to Ax_0 in measure.

Let A be a continuous and compact-in-measure operator acting from \mathcal{L}_p to \mathcal{L}_q ($1 \leq p, q \leq \infty$). In the general case the adjoint operator A^* , acting from $\mathcal{L}_{q'}$ to $\mathcal{L}_{p'}$, need not have the compactness-in-measure property. If, however, the operator A is regular and $1 \leq p < \infty$, $1 < q < \infty$, then the operator A is compact in measure if and only if the operator A^* is compact in measure. The regularity condition on the operator A can be replaced by the weaker assumption that A maps every set of functions with equiabsolutely continuous norms in \mathcal{L}_p into a set of functions with equiabsolutely continuous norms in \mathcal{L}_q .

In (3) it is shown that every regular integral operator acting from \mathcal{L}_p into \mathcal{L} is completely continuous if $1 < p \leq \infty$. From Theorem 3 it follows that every integral operator acting from \mathcal{L}_p into \mathcal{L}_q , where $1 < p \leq \infty$, $0 < q \leq 1$, is completely continuous. From these assertions it follows that every regular integral operator is compact in measure. We do not know whether there exist nonregular integral operators which do not have the property of compactness in measure. Let us note that an integral operator A is compact in measure if

$$\lim_{\mu_1(D) \rightarrow 0} \sup_{\|x\| \leq 1} |AP_{Dx}| = 0, \quad (4)$$

where

$$|z| = \inf_{\alpha} \{ \alpha + \mu_2[t : |z(t)| \geq \alpha] \}.$$

In particular, condition (4) is satisfied if the operator A can be extended to a continuous operator acting from some \mathcal{L}_{p_1} , where $p_1 < p$, into some \mathcal{L}_{q_1} .

The property of complete continuity of linear operators acting from \mathcal{L}_p into \mathcal{L}_q , $q < \infty$, may be regarded as a combination of the property of compactness in measure and the property of improvability (or, if $1 < p < \infty$, $1 \leq q \leq \infty$, as a combination of the property of compactness in measure of the adjoint operator and the property of co-improvability).

Theorem 5. *A linear operator A acting from \mathcal{L}_p into \mathcal{L}_q is completely continuous if it has the Ando property and if one of the following conditions is fulfilled:*

- a) A is regular and compact in measure, $q < \infty$;
- b) A is regular, A^* is compact in measure, $1 < p < \infty$, $1 \leq q \leq \infty$;
- c) A and A^* are compact in measure, $1 < p$, $q < \infty$.

The complete continuity of a regular integral operator acting from \mathcal{L}_p into \mathcal{L}_q , where $1 < q < p < \infty$, was proved in ⁽⁶⁾ by T. Ando (he also considered operators acting in Orlicz spaces).

4. Every norm-bounded sequence $x_n \in \mathcal{L}_p$ converging in measure converges weakly if $p > 1$. Therefore operators A acting from \mathcal{L}_p into \mathcal{L}_q ($1 < p < \infty$, $1 \leq q \leq \infty$) which are compact in measure transform norm-bounded sequences $x_n \in \mathcal{L}_p$ converging in measure to x_0 into sequences $Ax_n \in \mathcal{L}_q$ which also converge in measure.

Theorem 6. *Let A be an operator compact in measure, acting in \mathcal{L}_p ($1 < p < \infty$).*

Then the operator $T = I - A$ transforms every norm-bounded and measure-closed set $\mathfrak{M} \subset \mathcal{L}_p$ into a set $T\mathfrak{M}$ closed in measure.

It follows from Theorem 6 that the operator $(I - A)^{-1}$, if it exists, is representable in the form $I - B$, where B is an operator compact in measure.

5. In applying the theorems of the present article one should keep in mind that there are no nonzero linear continuous operators acting from \mathcal{L}_p into \mathcal{L}_q if $p < 1$, $p < q$. There do not exist nonzero linear integral operators with nonnegative kernels acting from \mathcal{L}_p into \mathcal{L}_q if $p < 1$. We do not know whether there exist nonzero completely continuous operators acting from some \mathcal{L}_p , where $p < 1$, into some \mathcal{L}_q .

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CITED LITERATURE

1. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
2. M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, 1959.
3. M. A. Krasnosel' skii, E. I. Pustyl' nik, DAN, 142, No. 1 (1962).
4. E. I. Pustyl' nik, DAN, 146, No. 6 (1962).
5. P. P. Zabreiko, E. I. Pustyl' nik, UMN, 19, issue 2 (1964).
6. T. Ando, Proc. K. Nederl. Akad. Wetensch., Ser. A, 65, No. 2, 235 (1962).
7. L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Semioordered Spaces*, 1950.
8. I. A. Bakhtin, M. A. Krasnosel' skii, V. Ya. Stetsenko, Sibirsk. Mat. Zhurn., 3, No. 1 (1962).

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