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Abstract

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A. M. MIKENBERG

ON COMPLETE SYSTEMS OF NONDECREASING GENERAL-RECURSIVE FUNCTIONS

(Presented by Academician P. S. Novikov on 24 IX 1963)

In the present note some questions concerning the study of properties of systems of nondecreasing general-recursive functions are elucidated.

Definition 1. A general-recursive function $f(n)$ is said to be **greater** than a general-recursive function $\varphi(n)$ if, for all natural numbers except for a finite number of them, the relation $f(n) \geq \varphi(n)$ holds. A general-recursive function $\varphi(n)$ is said to be **less** than a general-recursive function $f(n)$ if, for all natural numbers except for a finite number of them, the relation $\varphi(n) \leq f(n)$ holds. Two general-recursive functions are called **comparable** if one of them is greater or less than the other.

Definition 2. A set of nondecreasing general-recursive functions is called a **complete system** if all the functions of this set are comparable with one another, and every nondecreasing general-recursive function not belonging to the given set is not comparable with at least one nondecreasing general-recursive function of this set.

From these definitions the following properties of complete systems follow trivially.

Constant functions belong to a complete system.

Through every integer point of the coordinate plane there passes the graph of at least one nondecreasing general-recursive function of the complete system, distinct from a constant function.

For every nondecreasing general-recursive function of a complete system, there exists in this complete system a nondecreasing general-recursive function greater than it.

For every nondecreasing general-recursive function distinct from a constant function, there exists in this complete system a nondecreasing general-recursive function less than the given one and distinct from a constant function.

A complete system contains a countable set of nondecreasing general-recursive functions distinct from constant functions. Here by a constant function is meant a function equal to a constant number for all natural numbers except for a finite number of these numbers.

There do not exist complete systems of nondecreasing general-recursive functions consisting solely of primitive-recursive functions.

Definition 3. A set of partial-recursive functions $\{f(n)\}$ is called **computable** if there exists a partial-recursive function $F(t, n)$ that is universal for the given set of partial-recursive functions $\{f(n)\}$, and only for this set of functions.

Remark. If the set $\{f(n)\}$ consists only of general-recursive functions, then the function $F(t, n)$ is general-recursive.

Theorem 1. *A set $\{F(n)\}$ of partial-recursive functions is computable if and only if there exists a recursive set consisting of numbers of partial-recursive functions with respect to some partial-recursive numbering of all partial-recursive*

functions. This set contains at least one index of every partially recursive function from the set $\{f^{(n)}\}$.

Remark. It would be interesting to know whether there exist computable sets of partially recursive functions such that the set of all indices of the functions of the computable set is a recursive set.

Theorem 2. For every computable set of general recursive functions there exists a general recursive function greater than any function of this computable set; moreover, this function can be chosen so as to be increasing.

Theorem 3. For every computable set of nondecreasing general recursive functions, distinct, beginning with some argument, from constant functions, there exists a general recursive nondecreasing function smaller than each function of this computable set and distinct from a constant function, beginning with some argument.

Theorem 4. For every computable set of nondecreasing general recursive functions, distinct, beginning with some argument, from constant functions, there exists a nondecreasing general recursive function incomparable with each of the functions of the given computable set.

Remark. Theorems 2–4 do not carry over to computable sets of partially recursive functions. There exist partially recursive functions that are incomparable with every general recursive function, except for functions equal to zero for all arguments beginning with some natural number. If a partially recursive function is monotone, or if its domain of definition is a general recursive set, then such a function is majorized by a nondecreasing general recursive function. There exist computable sets of partially recursive functions that are not majorized by partially recursive functions. There is no partially recursive function that would majorize all general recursive functions. The question arises whether it is possible to majorize a partially recursive function by a function whose graph is an extension of a recursive set; whether it is possible for such a function to majorize all general recursive functions, all partially recursive functions.

Theorem 5. There exists a countable set of complete systems.

Theorem 6. There do not exist computable complete systems.

Corollary. The set of all indices of functions of a complete system, with respect to any partially recursive enumeration of all partially recursive functions, is not a recursive set.

Remark. The corollary remains valid if the set of indices of functions of a complete system is taken to include not all indices of each function of the complete system, but an arbitrary nonempty set of indices of each function of the complete system.

Definition 4. We shall call a predicate **arithmetical** if it can be obtained by binding the free variables of a general recursive predicate with the quantifiers "there exists" and "for all." A set is called **arithmetical** if there exists an arithmetical predicate whose truth set is this set. Suppose that some partially recursive enumeration of all partially recursive functions is given. A complete system is called **arithmetical** if the set of all indices of functions of the complete system with respect to the given enumeration is an arithmetical set.

Theorem 7. There exist arithmetical complete systems.

Remark. In the proof of Theorem 7 a complete system is constructed such that the set of indices of the functions of the complete system is the truth set of an arithmetical predicate with four quantifiers. Can the number of quantifiers of such a predicate be reduced?

Definition 5. A complete system is called **unbounded above** if for every nondecreasing general recursive function there exists in this complete system a nondecreasing general recursive ···

function greater than the given one. A complete system is called **unbounded below** if for every nondecreasing general-recursive function, distinct from a constant one from some argument onward, there exists in this complete system a nondecreasing general-recursive function smaller than the given one and distinct from a constant one from some argument onward.

Theorem 8. There exists an arithmetic complete system unbounded above.

Theorem 9. There exists an arithmetic complete system not bounded below.

Theorem 10. There exists an arithmetic complete system unbounded both above and below.

Theorem 11. There exists an arithmetic complete system for which there is such a nondecreasing general-recursive function that the complete system contains no nondecreasing general-recursive function greater than it.

Theorem 12. There exists an arithmetic complete system for which there is such a general-recursive nondecreasing function that the complete system contains no nondecreasing general-recursive function smaller than it and distinct from a constant function from some argument onward.

Theorem 13. There exists an arithmetic complete system for which there is such a general-recursive nondecreasing function that it is incomparable with all functions of the complete system, except for functions that are constant from some argument onward.

Moscow State Pedagogical Institute
named after V. I. Lenin

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REFERENCES

1. S. C. Kleene, *Introduction to Metamathematics*, Moscow, 1959.
2. P. S. Novikov, *Elements of Mathematical Logic*, Moscow, 1959.
3. R. Peter, *Recursive Functions*, Moscow, 1954.
4. V. A. Uspensky, *Lectures on Computable Functions*, Moscow, 1960.

Note: Figure translations are in progress. See original paper for figures.