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Abstract

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On the Causes of the Formation of Shock Waves in Laval Nozzles

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In work ([1]), using the example of one particular solution of the system of equations describing transonic gas flows, the causes leading to the formation of shock waves in the vicinity of the throat of a Laval nozzle were analyzed. Below, with the same aim, a more general class of solutions of the equations of transonic flows is considered:

$$-u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{1}$$

Here x and y are axes of dimensionless Cartesian coordinates, and the dimensionless functions u and v are proportional to the components, along these axes, of the vector of the disturbed velocity of the particles ([2]).

The discontinuous solutions of the system of equations (1) that are to be studied are obtained as a result of solving the following Cauchy problem.

Let, for $y = 0$, i.e., on the axis of symmetry of the flow,

$$u = -A_1|x|^k \quad \text{for } x < 0; \qquad u = A_2 x^k \quad \text{for } x > 0, \quad v = 0 \tag{2}$$

$$(A_1 > 0,\ A_2 > 0).$$

We shall assume that the values of the exponent k lie in the interval $1 < k < 2$; in the corresponding gas motions the sonic line is a power curve concave toward the oncoming flow ([3]).

Discontinuous solutions of the system of equations (1) describe flows with compression jumps. Such solutions must satisfy, in addition to the initial data (2), additional boundary conditions at the wave front: the shock polar equation ([4])

$$2(v_2 - v_3)^2 = (u_2 - u_3)^2(u_2 + u_3) \tag{3}$$

and the relation ([4])

$$u_2\frac{dx_2}{dy}+v_2=u_3\frac{dx_2}{dy}+v_3, \tag{4}$$

which follows from the condition of continuity of the tangential component of the velocity vector. In equalities (3) and (4), the indices refer to quantities on different sides of the shock front, and $x_2 = x_2(y)$ is the equation specifying its position.

It is easy to show that the desired solution of the Cauchy problem is self-similar:

$$u = y^{2(n-1)} f(\xi), \qquad v = y^{3(n-1)} g(\xi), \qquad \xi = x/y^n, \qquad n = 2/(2-k),$$

where the equation of the shock front has the form $\xi = \xi_2 = \text{const}$.

Substitution of the written formulas into equation (1) and elimination of the function g from the resulting relations give, for determining f, the second-order differential equation

$$(f-n^2\xi^2)\frac{d^2f}{d\xi^2}+\left(\frac{df}{d\xi}\right)^2+n(3n-5)\xi\frac{df}{d\xi}-2(2n-3)(n-1)f=0. \tag{5}$$

To simplify the qualitative investigation of the problem under consideration, let us set [(1)]

$$f=\xi^2F(\eta), \qquad dF/d\eta=\Psi, \qquad \eta=\ln|\xi|.$$

In the new variables the order of equation (5) is reduced:

$$\frac{d\Psi}{dF}=\frac{-6F-5n\Psi+6F^2+7F\Psi+\Psi^2}{(n^2-F)\Psi}. \tag{6}$$

The initial data (2) lead to the requirement that the integral curve of equation (6), which gives the flow field in the neighborhood of the nozzle throat, begin and end at its singular point $A(0,0)$, which corresponds to the x-axis. In the neighborhood of A this curve is determined by the expansion

$$\Psi=-\frac{2}{n}F-2\frac{3n^2-7n+4}{n^3}F^2+2\frac{(4n^2-7n+2)(3n^2-7n+4)}{n^5}F^3+\ldots \tag{7}$$

As for the boundary conditions (3) and (4), in the $F\Psi$-plane they take the form

$$F_2+F_3=2n^2, \qquad \Psi_2+\Psi_3=-2n(7n-5). \tag{8}$$

Motion along the integral curve in the $F\Psi$-plane from the point A in the direction of the singular point $C(n^2, -n(n+1))$ represents the gas flow in the inlet part of the nozzle between the x-axis and the C_-^0-characteristic arriving at its center. Passing through the point C means intersection of the C_-^0-characteristic in the physical plane. In order to construct a flow free of shock waves in the region between the C_-^0-characteristic arriving at the center of the nozzle and the C_+^0-characteristic leaving it, one may move from the point C along any integral curve in the direction of the infinitely distant point E, which lies on the straight line $\Psi = -2F$, and then return along the continuation of this curve again to the point C. If the continuation of the curve issuing from C is the unique integral curve of equation (6) passing through the singular point $D(n^2, -6n(n-1))$, then in the physical plane we obtain the flow which, among all continuous flows, expands most slowly downstream of the C_+^0-characteristic. A jump from the point C to the point D and subsequent motion along the indicated curve in the reverse direction through E again to the point C represents the flow which expands most rapidly downstream of the C_+^0-characteristic.

In the first of these two limiting continuous flows, discontinuities of acceleration are formed on the C_+^0-characteristic; in the second, on the C_-^0-characteristic. In all other continuous flows, along the indicated characteristics only higher derivatives of the velocity components are discontinuous.

In the discontinuous gas flows under consideration, the shock is born at the center of the channel and is then carried downstream. To construct discontinuous flows it is necessary to choose integral curves of equation (6) which leave the point C in the direction of the point E and whose continuations, beginning at E, lie below the unique integral curve passing through the point D. Along such curves, as $F \to n^2$ the values $\Psi \to -\infty$, and in the corresponding gas flows limiting lines arise that carry infinite values of the accelerations. Since a flow with infinite accelerations is physically meaningless, a shock wave must form in it before the limiting line appears. It is important to emphasize that a shock wave is formed only when a limiting line has first appeared in the flow; it is impossible to introduce a shock wave into a flow in which there are no infinite accelerations. We note that in the inlet part of the nozzle the shock does not disturb the motion of the gas.

The gas flow behind the compression shock in the variables F, Ψ must be represented by a segment of an integral curve (7). This condition, together with the equalities (8), determines the intensity of the shock wave. For computations it is simplest to use direct integration of the original equation (5), then transferring the results to the $F\Psi$-plane. In this way one automatically obtains the coordinate ξ_2, which gives the position of the shock front. In addition, for part of the computations it is also convenient to use S. A. Chaplygin' s hodograph method, which makes it possible to pass from the nonlinear system of equations (1) to Tricomi' s linear equation for the function $y(u, v)$

Fig. 1

Fig. 1

Figure 1: Fig. 1

$$\frac{\partial^2 y}{\partial u^2} - u\frac{\partial^2 y}{\partial v^2} = 0. \tag{9}$$

The solution of equation (9) satisfying the initial data (2) in the subsonic part of the flow, in a neighborhood of the axis $v = 0$, can be written in the form

$$y = -4^{1/2-j}(kA_1^{1/k})^{-1}v(9v^2 - 4u^3)^{j-1/2}\times$$

$$\times F\left(\frac{1}{2} - j,\ \frac{2}{3} + j,\ \frac{3}{2};\ \frac{9v^2}{9v^2 - 4u^3}\right), \tag{10}$$

$$j = (2-k)/(6k),$$

where F, as usual, denotes the hypergeometric function. In the supersonic part of the flow, in a neighborhood of the axis of symmetry of the flow, the solution of equation (9) can be represented in the form

$$y = (kA_2^{1/k})^{-1}vu^{3j-3/2}F\left(\frac{1}{2} - j,\ \frac{5}{6} - j,\ \frac{3}{2};\ \frac{9v^2}{4u^3}\right). \tag{11}$$

The self-similar solutions of (9) written out above were first considered by Tricomi ([5]); their connection with the theory of the Laval nozzle was established in ([3]). The form of the solution in a neighborhood of the transition line $u = 0$ and of the characteristics $v = \mp\frac{2}{3}u^{3/2}$ is obtained from equalities (10) and (11) by means of formulas for analytic continuation of hypergeometric functions. Throughout the entire region of gas motion the Jacobian $D(x,y)/D(u,v)$ must retain a negative sign.

As a result of computations based on the hodograph method, we arrive at the inequalities

$$\frac{1}{\left[4\sin\pi\left(\frac{2}{3} - j\right)\sin\pi\left(\frac{1}{6} + 2j\right)\right]^k} \le \frac{A_2}{A_1} \le \left[\frac{\sin\pi\left(\frac{1}{6} + 2j\right)}{\sin\pi\left(\frac{2}{3} - j\right)}\right]^k,$$

the fulfillment of which is sufficient for the shock-free character of the flow. Conversely, if

Fig. 2

Figure 2: Fig. 2

$$0 < \frac{A_2}{A} < \frac{1}{\left[4 \sin \pi \left(\frac{2}{3} - j\right) \sin \pi \left(\frac{1}{6} + 2j\right)\right]^k}, \tag{12}$$

then in the corresponding gas flows a shock wave is formed, on both sides of which the velocity is supersonic. The latter inequalities also ensure a further increase in velocity and expansion of the flow in the region behind the compression shock; however, this expansion occurs more slowly than in continuous flows. For $A_2 = 0$, in the entire region behind the shock front there is realized a uniform sonic flow moving parallel to the x-axis. We note that in all the gas motions constructed, $A_2 < A_1$.

In both the continuous and discontinuous flows considered above, along the C_-^0-characteristic closing the inlet part of the nozzle, generally speaking, certain singularities propagate in the derived components

velocities with respect to the coordinates. From an analysis of (6) it follows that, for $1 < k < {}^4/_3$, all derivatives of the velocity components, beginning with the third, have infinite discontinuities; when ${}^4/_3 < k < {}^8/_5$, the fourth and higher derivatives have infinite discontinuities; for ${}^8/_5 < k < {}^{20}/_{11}$, infinite discontinuities arise in the fifth and higher derivatives; finally, when ${}^{20}/_{11} < k < 2$, all derivatives of the components of the velocity vector, beginning with the sixth, undergo infinite discontinuities. For $k = {}^4/_3, {}^8/_5$, and ${}^{20}/_{11}$, on the C_-^0-characteristic the third, fourth, and fifth derivatives, respectively, have finite discontinuities. Only two types of flows can be constructed that are free of any singularities on the C_-^0-characteristic; these flows, like the well-known flow of F. I. Frankl [7] and S. V. Falkovich [8], have an asymptotic character in a neighborhood of the center of the nozzle [6]. All the other flows of the class under consideration can be realized only in nozzles of a special shape, whose walls at the points of intersection with the C_-^0-characteristic have quite definite singularities.

Fig. 2

The dependence $f_3 - f_2$ on the constant A_2, shown in Fig. 1 as an example for $k = {}^{20}/_{11}$, characterizes the intensity of the shock wave. For this value of k, one of the discontinuous solutions has no singularities in a neighborhood of the C_-^0-characteristic. The quantity A_1 is chosen so that its position is determined by the equality $\xi_1 = -1$. Figure 2 gives the dependence of the coordinate ξ_2 of the compression shock on the constant A_2. The points in Figs. 1 and 2 mark the quantities corresponding to the gas flow analytic in the inlet part of the nozzle [6].

It follows from formula (12) that shock waves in flows are formed only when the values of the ratio A_2/A_1 become lower than a certain limit. It is easy to show that, as A_2/A_1 decreases, the streamlines in the region beyond the C_-^0-characteristic become ever more gently sloping; therefore, in nozzles in which shock waves arise, the distance between the throat and the inlet part of the channel is considerably greater than in nozzles that yield continuous gas flows. An excessively elongated transition part of the channel is the cause of the formation of shock waves in the vicinity of its critical section. In designing nozzles, the transition part should be made as short as possible. The same conclusions were obtained in [1].

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Note: Figure translations are in progress. See original paper for figures.