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THEORY OF ELASTICITY

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## Abstract

## Full Text

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### *THEORY OF ELASTICITY*

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## TRANSVERSE BENDING OF AN ISOTROPIC PLANE RESTING ON A DOUBLY PERIODIC SYSTEM OF POINT SUPPORTS

The investigation of the bending of an isotropic plane resting on a doubly periodic system of point supports and subjected to a uniform transverse load is contained in works <sup>(1,2)</sup> and others. The solutions obtained in these works cannot give information on the concentration of stresses near the supports. In the present paper, on a fundamentally different basis, a solution of the problem is constructed that is free of the indicated drawback.

1. Let  $\omega_1 = 2$ ;  $\omega_2 = 2le^{i\alpha}$  ( $l > 0$ ) be the fundamental periods. We assume the congruent system of point supports  $P = m\omega_1 + n\omega_2$  ( $m, n = 0, \pm 1, \pm \dots$ ) to be symmetric with respect to the coordinate axes  $x$  and  $y$ . The origin of coordinates is taken to coincide with the point  $P = 0$ .

We represent the general solution of the differential equation of bending

$$\nabla^2 \nabla^2 w(x, y) = \frac{q}{D} \quad (1)$$

in the form

$$w(x, y) = \frac{q(z\bar{z})^2}{64D} + \frac{2q}{D} \operatorname{Re}\{\bar{z}\varphi(z) + \chi(z)\}. \quad (2)$$

Here  $w(x, y)$  is the deflection function;  $q$  and  $D$  are the intensity of the uniform load and the cylindrical rigidity of the plate;  $\varphi(z)$  and  $\chi(z)$  are arbitrary analytic functions.

The problem will obviously be solved if the functions  $\varphi(z)$  and  $\chi(z)$  are constructed in such a way as to satisfy the conditions of double periodicity of the deflections, symmetry with respect to the coordinate axes, and the condition

$$w(0) = 0. \quad (3)$$

To construct the functions  $\varphi(z)$  and  $\chi(z)$ , we introduce the system of functions

$$\begin{aligned}\varphi(z) &= -\zeta'(z) = -\nu''(z) = -\xi'''(z), \\ Q(z) &= -\zeta'_*(z) = -\nu''_*(z) = -\xi'''_*(z),\end{aligned}\tag{4}$$

where

$$\begin{aligned}\varphi(z) &= \frac{1}{z^2} + \sum'_{m,n} \left\{ \frac{1}{(z-P)^2} - \frac{1}{P^2} \right\}, \\ Q(z) &= \sum'_{m,n} \left\{ \frac{\bar{P}}{(z-P)^2} - 2z \frac{\bar{P}}{P^3} - \frac{\bar{P}}{P^2} \right\};\end{aligned}$$

$\varphi(z)$  is the Weierstrass elliptic function <sup>(3)</sup>, and  $Q(z)$  is a special meromorphic function.

The identities <sup>(4)</sup> hold:

$$\begin{aligned}Q(z + \omega_1) - Q(z) &= \bar{\omega}_1 \varphi(z) + \gamma_1, \\ Q(z + \omega_2) - Q(z) &= \bar{\omega}_2 \varphi(z) + \gamma_2,\end{aligned}\tag{5}$$

where

$$\gamma_2 \omega_1 - \gamma_1 \omega_2 = \delta_1 \bar{\omega}_2 - \delta_2 \bar{\omega}_1,\tag{6}$$

$$\delta_1 = \zeta(z + \omega_1) - \zeta(z) = 2\zeta\left(\frac{\omega_1}{2}\right), \quad \delta_2 = \zeta(z + \omega_2) - \zeta(z) = 2\zeta\left(\frac{\omega_2}{2}\right);$$

$\zeta(z)$  is the Weierstrass zeta function.

Indeed, differentiating (5) and substituting the expressions  $Q'(z + \omega_1)$ ,  $Q'(z)$ , and  $\varphi'(z)$  in explicit form, we verify the validity of the identities

$$\begin{aligned}Q'(z + \omega_1) - Q'(z) &= \bar{\omega}_1 \varphi'(z), \\ Q'(z + \omega_2) - Q'(z) &= \bar{\omega}_2 \varphi'(z),\end{aligned}\tag{7}$$

whence (5) follows. Relation (6) is easily obtained by considering the integral  $\int Q(z) dz$  along the contour of the parallelogram with vertices  $0.5(\omega_1 + \omega_2)$ ;  $0.5(\omega_2 - \omega_1)$ ;  $-0.5(\omega_1 + \omega_2)$ , and  $0.5(\omega_1 - \omega_2)$ .

Differentiating (5) and taking (4) into account, we obtain the relations

$$\xi_*(z + P) - \xi_*(z) = \bar{P}\xi(z) + R_3(z),$$

$$\xi(z + P) - \xi(z) = R_2(z), \quad P = m\omega_1 + n\omega_2, \quad (m, n = 0, \pm 1, \dots), \quad (8)$$

where  $R_k(z)$  is a polynomial of degree  $k$  with coefficients depending on the fundamental periods  $\omega_1$  and  $\omega_2$ .

2. Put in (2)

$$\begin{aligned} \varphi(z) &= A_1z + A_2z^3 + A_3\xi(z), \\ \chi(z) &= B_1z^2 + B_2z^4 - A_3\xi_*(z), \end{aligned} \quad (9)$$

and seek the coefficients in (9) from the condition of double periodicity (2). The fundamental possibility of satisfying this condition follows from the equalities (8). Representing the conditions of double periodicity of the deflections in the form

$$\begin{aligned} w(z + \omega_1) - w(z) &= 0, \\ w(z + \omega_2) - w(z) &= 0, \end{aligned} \quad (10)$$

we find, taking account of (8), (6), (5), (4), and Legendre's relation  $\delta_1\omega_2 - \delta_2\omega_1 = 2\pi i$  (3),

$$\begin{aligned} A_3 &= \frac{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2}{32\pi i}, \\ A_2 &= \frac{1}{96} \left\{ \frac{\delta_1(\bar{\omega}_1\omega_2 - \omega_1\bar{\omega}_2)}{2\pi i \omega_1} - 1 \right\}, \\ B_2 &= -\frac{\bar{A}_2\bar{\omega}_1}{4\omega_1} - \frac{A_3\gamma_1}{24\omega_1}. \end{aligned} \quad (11)$$

The constants  $A_1$  and  $B_1$  are determined from the system

$$\begin{aligned} 2A_1\bar{\omega}_1 + 2B_1\omega_1 &= A_3K_1(\omega), \\ 2A_1\bar{\omega}_2 + 2B_1\omega_2 &= A_3K_2(\omega), \end{aligned} \quad (12)$$

where

$$K_1(\omega) = 2\nu_*\left(\frac{\omega_1}{2}\right) - \bar{\omega}_1\nu\left(\frac{\omega_1}{2}\right) - 2\xi\left(\frac{\omega_1}{2}\right) + \frac{\gamma_1\omega_1^2 + 2\delta_1\omega_1\bar{\omega}_1 + \bar{\delta}_1\bar{\omega}_1^2}{24},$$

$$K_2(\omega) = 2\nu_*\left(\frac{\omega_2}{2}\right) - \bar{\omega}_2\nu\left(\frac{\omega_2}{2}\right) - 2\xi\left(\frac{\omega_2}{2}\right) + \frac{\gamma_2\omega_2^2 + 2\delta_2\omega_2\bar{\omega}_2 + \bar{\delta}_2\bar{\omega}_2^2}{24}.$$

The quantities  $A_i$  and  $B_i$ , as is not difficult to verify, are real.

Integrating  $\wp(z)$  and  $Q(z)$ , according to (4), one may find representations of all the functions of interest to us. Thus,

$$\begin{aligned} \xi(z) &= z \ln z - z + \sum'_{m,n} \left\{ (z-P) \ln \left(1 - \frac{z}{P}\right) - z + \frac{z^2}{2P} + \frac{z^3}{6P^2} \right\}, \\ \xi_*(z) &= \sum'_{m,n} \left\{ \bar{P}(z-P) \ln \left(1 - \frac{z}{P}\right) - \bar{P}z + \frac{z^2\bar{P}}{2P} + \frac{z^3\bar{P}}{6P^2} + \frac{z^4\bar{P}}{12P^3} \right\}. \end{aligned} \quad (13)$$

From (2), taking into account (8), (9), (11), and (13), it is immediately seen that condition (3) is satisfied. Thus, the expression for the deflection function (2), taking into account (9), (11), and (12), gives the solution of the posed problem.

Differentiating (2), we find expressions for the moments and transverse forces

$$\begin{aligned} M_x + M_y &= -q(1 + \mu) \left\{ \frac{z\bar{z}}{4} + 8 \operatorname{Re} \varphi'(z) \right\}, \\ M_y - M_x + 2iH_{xy} &= q(1 - \mu) \left\{ \frac{\bar{z}^2}{8} + 4\bar{z}\varphi''(z) + 4\chi''(z) \right\}, \\ N_x - iN_y &= -\frac{q}{2}\bar{z} - 8q\varphi''(z), \end{aligned} \quad (14)$$

where  $M_x$ ,  $M_y$ , and  $H_{xy}$  are the bending and twisting moments on the corresponding elements;  $N_x$  and  $N_y$  are the transverse forces.

From formulas (14), taking into account (9), (4), (13), and (11), we find representations of the moments and transverse forces in a sufficiently small neighborhood of the support point

$$P = m\omega_1 + n\omega_2 \quad (m, n = 0, \pm 1, \dots)$$

$$M_x = M_y = \frac{1 + \mu}{4\pi} T \ln |z - P| - 4q(1 + \mu)A_1,$$

$$N_x - iN_y = \frac{T}{2\pi} \frac{1}{z - P}, \quad T = qs, \quad (15)$$

where  $s$  is the area of the parallelogram of periods;  $\mu$  is Poisson's ratio.

Formulas (15) give information about the stress concentration in an arbitrarily small neighborhood of the support.

For the cases when the system of point supports is regular, i.e., either regular triangular ( $\omega_1 = 2; \omega_2 = 2e^{i\pi/3}$ ), or square ( $\omega_1 = 2; \omega_2 = 2i$ ), the formulas are substantially simplified. We give the values of the maximum deflections for the indicated special cases:

$$\text{a) } \omega_1 = 2; \omega_2 = 2e^{i\pi/3}; \quad \frac{\pi^4 D}{q} w_{\max} = 5.80;$$

$$\text{b) } \omega_1 = 2; \omega_2 = 2i; \quad \frac{\pi^4 D}{q} w_{\max} = 9.04.$$

In conclusion, we note that all the reasoning here was carried out for the normalized principal periods  $\omega_1 = 2$  and  $\omega_2 = 2le^{i\alpha}$ . If the periods have the form  $\omega_1 = a$  and  $\omega_2 = ale^{i\alpha}$ , then the deflection values must obviously be multiplied by the quantity  $(a/2)^4$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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