MATHEMATICS

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1964

SovietRxiv

Abstract

Full Text

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ON GENERALIZED SOLVABLE GROUPS AND THEIR GROUP RINGS

(Presented by Academician A. I. Mal' cev on 2 XII 1963)

1. A **primitive class** or **variety of groups** is a class of all groups satisfying some fixed system of identical relations.

Let a group G and some collection of words S in the unknowns $X = \{x_1, x_2, ...\}$ be given. Denote by $V(S, G)$ the subgroup of G generated by all values of the words from S in the group G, i.e.

$$V(S, G) = \{u(g_1, \dots, g_n)\},$$

where $u(x_1, \dots, x_n) \in S,\ g_1, \dots, g_n \in G$. $V(S, G)$ is called the **verbal subgroup** ([1]) of the group G generated by S. If the set S is empty, then we put $V(S, G) = 1$. The verbal subgroup of a group G determined by a variety of groups $\mathfrak{M}$ will be denoted by $V(\mathfrak{M}, G)$. If F is an arbitrary free group, then the factor group $F/V(\mathfrak{M}, F)$ is called a **free $\mathfrak{M}$-group**.

We shall call a variety of groups $\mathfrak{M}$ **normal** if, for every normal divisor A of any noncommutative free group F, the equality $V(\mathfrak{M}, A) = V(\mathfrak{M}, F)$ implies $A = F$.

Let some variety of groups $\mathfrak{M}$ be given. A normal system of a group G all of whose factors belong to the variety $\mathfrak{M}$ will be called a **normal $\mathfrak{M}$-system**.

If a normal system $\mathfrak{A} = \{A_\alpha\}$ of a group G contains a normal divisor A of this group, then the subsystem $\mathfrak{B}$ of the system $\mathfrak{A}$ consisting of all A_α that contain A (including A itself) shall be called a **normal system** of the group G relative to A. The factor groups of the subgroups from $\mathfrak{B}$ by the normal divisor A evidently form a normal system of the group G/A.

Theorem 1. *Let $\mathfrak{M}$ be a normal variety of groups, F a noncommutative free group, and A a normal divisor of the group F distinct from the identity. If the group F possesses a normal $\mathfrak{M}$-system $\mathfrak{S} = \{F_\alpha\}$ $(1 \le \alpha \le \gamma,\ F_1 = F,\ F_\gamma = A)$ relative to A, then there also exists a normal system $\mathfrak{S}^*$ of the group F relative to $V(\mathfrak{M}, A)$, all of whose factors are subgroups of free $\mathfrak{M}$-groups. The system $\mathfrak{S}^*$ may be obtained from the system*

$$\mathfrak{S}' = \{V_\alpha\} \quad (0 \leq \alpha \leq \gamma),$$

where $V_0 = F,\ V_\alpha = V(\mathfrak{M}, F_\alpha)\ (1 \leq \alpha \leq \gamma)$, *by supplementing the latter with the intersections of all subgroups of each of its subsystems.*

Fix some variety of groups $\mathfrak{M}$ and distinguish the following **types of normal systems in groups**:

$N_\mathfrak{M}$ —a normal $\mathfrak{M}$-system,

$I_\mathfrak{M}$ —an invariant $\mathfrak{M}$-system,

$N^*_\mathfrak{M}$ —a normal $\mathfrak{M}$-system, completely ordered by ascending order,

$I^*_\mathfrak{M}$ —an invariant $\mathfrak{M}$-system, completely ordered by ascending order,

$K_\mathfrak{M}$—a normal $\mathfrak{M}$-system, well ordered in descending order,
$R_\mathfrak{M}$—a finite normal series all of whose factors belong to the variety $\mathfrak{M}$.

We shall call a group G a T-group ($T = N_\mathfrak{M}, I_\mathfrak{M}, N^*_\mathfrak{M}, I^*_\mathfrak{M}, K_\mathfrak{M}, R_\mathfrak{M}$) if it has a normal system of type T. In particular, if $\mathfrak{M}$ is the variety of all abelian groups, then the classes $N_\mathfrak{M}$-, $I_\mathfrak{M}$-, $N^*_\mathfrak{M}$-, $I^*_\mathfrak{M}$-, $K_\mathfrak{M}$-, $R_\mathfrak{M}$-groups coincide respectively with the classes of RN-, RI-, RN^*-, RI^*-, RK-, R-groups ([2]). From Theorem 1 there follows immediately:

Corollary. Let $\mathfrak{M}$ be a normal variety of groups. If the factor group F/A of a noncommutative free group F by some nonidentity normal divisor A is a T-group ($T = N_\mathfrak{M}, I_\mathfrak{M}, N^*_\mathfrak{M}, I^*_\mathfrak{M}, K_\mathfrak{M}, R_\mathfrak{M}$), then the factor group $F/V(\mathfrak{M}, A)$ has a normal system of the same type T, all factors of which are subgroups of free $\mathfrak{M}$-groups.

If A, B are arbitrary normal divisors of a noncommutative free group F, then, as M. Auslander and Lyndon established ([3]), the equality $[A, A] = [B, B]$ implies $A = B$. Consequently, the variety of all abelian groups is normal and, in view of Theorem 1, the following is also true:

Theorem 2. If the factor group F/A of a free group F by some normal divisor A is a T-group ($T = RN, RI, RN^*, RI^*, RK, R$), then the factor group $F/[A, A]$ has a normal system of type T, all factors of which are free abelian groups.

Hence, in view of the results of M. I. Zaitseva ([4]) and A. A. Bovdi ([5]), there follows immediately:

Corollary. If the factor group F/A of a free group F by some normal divisor A is an RN-group, then the factor group $F/A^{(n)}$ of the group F by any n-th commutant $A^{(n)}$ of the subgroup A is right-orderable, and its group ring $P\left[F/A^{(n)}\right]$ over any field P contains no zero divisors.

Let a sequence of natural numbers $m_1, m_2, ...$, greater than 1, be given. In every group G there is a chain of normal divisors $G \supset G_{m_1} \supset G_{m_1,m_2} \supset \cdots$, where G_{m_1} is the m_1-th member of the lower central series of the group G, G_{m_1,m_2} is the m_2-th member of the lower central series of the group G_{m_1}, and so on. The group G is called polynilpotent of class $\mathfrak{m} = (m_1, \dots, m_k)$ if $G_{m_1,\dots,m_k} = 1$. The polynilpotent groups of any given class $\mathfrak{m} = (m_1, \dots, m_k)$ form, by a result of B. H. Neumann ([6]), a normal variety. Therefore Theorem 1 is applicable also in this more general situation.

From the work of H. Neumann ([7]) it follows that the variety of groups defined by the identities $x^p = 1$, $xy = yx$ (p a prime number) is also normal. The identity $x^n = 1$ ($n > 1$) also defines a normal variety of groups.

In the free semigroup of all varieties of groups distinct from the zero and unit varieties, the normal varieties form an isolated free subsemigroup.

2. A group G is called **radical** ([8]) if it has a normal system, well ordered in increasing order, with locally nilpotent factors. A group G which locally has this property is called **locally radical**. The class of locally radical groups contains, in particular, all locally solvable groups and all RN^*-groups.

Theorem 3. If G is an arbitrary locally radical torsion-free group, then the following three conditions are equivalent:

1. The integral group ring $Z[G]$ of the group G is isomorphically representable by matrices over some field.

II. For any free presentation $G \cong F/A$ of the group G, the factor group $F/[A, A]$ is isomorphically representable by matrices over some field.

III. G is a finite extension of an abelian group.

Corollary. If G is a locally radicable ordered group (or, in particular, any locally nilpotent torsion-free group), then each of conditions I, II in Theorem 3 is equivalent to the condition

III′. The group G is abelian.

In the proof of Theorem 3 one uses A. I. Mal' cev' s theorem on locally soluble matrix groups ([9], Theorem 1) and the following propositions.

Lemma 1. Let A be such a normal divisor of the free group F that F/A is a torsion-free group. Then every locally nilpotent subgroup H of the group $F/[A, A]$ is either contained in the group $A/[A, A]$, or is isomorphic to some subgroup of the additive group of rational numbers.

The proof of this lemma can be obtained from A. I. Mal' cev' s theorem ([10]) on commuting elements of the group $F/[A, A]$.

Lemma 2. If the integral group ring $Z[G]$ of the group $G = F/A$ (where F is a free group and A is some normal divisor of it) is isomorphically representable by matrices over some field P, then the factor group $F/[A, A]$ also admits a faithful representation by matrices over a purely transcendental extension of the field P.

Lemma 3. If the group G contains a subgroup H of finite index whose integral group ring $Z[H]$ is isomorphically representable by matrices over some field P of characteristic zero, then the integral group ring $Z[G]$ of the group G also admits a faithful representation by matrices over the same field.

Using Theorem 2 and Lemma 3, it is easy to construct an example of a non-commutative right-orderable group G whose group ring $Z[G]$ admits a faithful matrix representation.

Theorem 4. The free n-step soluble group G_n, for $n \geq 3$, and the integral group ring $Z[G_2]$ of the free two-step soluble group G_2 are not faithfully representable by matrices for any choice of the representation field. The free two-step soluble group G_2 is isomorphically representable by matrices over some purely transcendental extension of the field of rational numbers.

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Received
28 XI 1963

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Note: Figure translations are in progress. See original paper for figures.