S. I. Khudyaev

1964

SovietRxiv

Abstract

Full Text

S. I. Khudyaev

ON BOUNDARY-VALUE PROBLEMS FOR CERTAIN QUASILINEAR ELLIPTIC EQUATIONS

(Presented by Academician I. N. Vekua, 9 IX 1963)

The work is devoted to the study of the basic boundary-value problems (the Dirichlet problem and the Neumann problem) of the stationary theory of thermal explosion ([6,7]). Questions of solvability, the qualitative behavior of solutions as functions of the main parameters, and also questions of local uniqueness and stability of solutions with respect to "small perturbations" are studied. Examples of nonuniqueness in the large are contained in ([6,8,9]).

In ([5]), and also in ([10]), a solvability criterion was established for the Dirichlet problem. An analogous criterion is established in the present work for the Neumann problem. These theorems in the present work serve as the principal tool in the study of boundary-value problems. We shall consider problems in a more general form than that in which they occur in the theory of thermal explosion.

I. Let, in a bounded domain G of n-dimensional space, the equation

$$L(u) \equiv \sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i(x) \frac{\partial u}{\partial x_i} + a(x, u) = 0 \tag{1}$$

be given, and on the boundary S one of the following boundary conditions:

$$M(u) \equiv \frac{du}{dT_x} + \sigma(x) u = \varphi(x); \tag{2}$$

$$u = f(x); \tag{3}$$

the transversal derivative at the point $x \in S$ is

$$\frac{du}{dT_x} = \lim_{y \to x} \sum_{ij} a_{ij}(y) \times$$

$$\times \frac{\partial u}{\partial y_i} \cos(\mathbf{n}_x, x_i),$$

where $\mathbf{n}_x$ is the vector of the outward normal to the boundary S at the point x; $(\mathbf{n}_x, x_i)$ is the angle between the normal and the i-th coordinate axis.

Theorem 1. Suppose the following conditions are satisfied:

A. There exist functions $\underline{u}, \overline{u} \in C^2(\overline{G})$ such that:

1. $\underline{u} \leq \overline{u}$.
2. $L(\underline{u}) \geq 0 \geq L(\overline{u})$.
3. $M(\underline{u}) \leq \varphi(x) \leq M(\overline{u})$.

B. $a_{ij}(x) \in C^\alpha(G)$; $a_i \in C^\alpha(G)$; $a(x,u)$ is defined in the domain

$$G_1 = \{x \in G;\ \underline{u}(x) \leq u \leq \overline{u}(x)\}$$

and, together with a_u, belongs to $C^\alpha(G_1)$, $\alpha < 1$.

C. The ellipticity condition is satisfied:

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_i \xi_i^2, \qquad a_0 > 0.$$

D. $S \in A^{1+\alpha}$ (see ([1])); $\sigma(x)$ and $\varphi(x)$ are continuous on S, and $\sigma(x) \geq 0$.

Under these conditions there exists at least one solution $u(x)$ of problem (1)–(2), belonging to $C^\theta(\overline{G})$ for any $0 < \theta < 1$ and to $C^{2+\beta}(G')$ for any interior subdomain $G' \subset G$, with $0 < \beta < \alpha$. Moreover, the solution $u(x)$ satisfies the inequalities

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x). \tag{4}$$

The **proof of Theorem 1** can be carried out analogously to how this is done in ([10]) in the case of the Dirichlet problem. Using

results of the works ([2–4]), one can show that, as $t \to \infty$, the solution $u(x,t)$ of the problem

$$\frac{du}{dt} = L(u); \qquad u\Big|_{t=0} = \underline{u}; \qquad M(u) = \varphi(x) \text{ on } S \tag{5}$$

converges to the solution of problem (1)–(2), possesses the required smoothness, and satisfies inequalities (4).

Corollary 1. *It is easy to see that the constructed solution $u(x)$ is the minimal solution satisfying inequalities (4). It is also obvious that the solution of problem (5) with initial condition $u\big|_{t=0} = \overline{u}$ would lead us to the maximal solution. The same is true for problem (1), (3).*

Let $a(x,u)$ be defined for $u \geqslant 0$ and $a(x,0) > 0$. Let $\varphi(x) \geqslant 0$. Then for any function $\sigma(x) \geqslant 0$, as the lower function $\underline{u}(x)$ one may take $\underline{u} \equiv 0$.

Corollary 2. *If problem (1)–(2) is solvable for some function $\sigma_0(x)$, then it is solvable for any $\sigma(x) \geqslant \sigma_0(x)$, and there exists a solution $u_\sigma(x)$ decreasing with respect to σ.*

If, for $\sigma_0(x)$, problem (1)–(2) is not solvable, then it is not solvable for any $\sigma(x) \leqslant \sigma_0(x)$.

We shall consider in equation (1), instead of $a(x,u)$, the function $\lambda a(x,u)$ for $\lambda \geqslant 0$, and assume additionally that $a(x,u) \geqslant 0$ and, in condition (3), $f(x) \geqslant 0$.

Corollary 3. *If problem (1)–(2) [(1), (3)] is solvable for some $\lambda_0 > 0$, then it is solvable for any $\lambda \leqslant \lambda_0$, and there exists a solution $u_\lambda(x)$ increasing with respect to λ.*

If, for $\lambda_0 \geqslant 0$, problem (1)–(2) [(1), (3)] is not solvable, then it is not solvable for any $\lambda \geqslant \lambda_0$.

For problem (1)–(2) [(1), (3)] the following cases are possible a priori:

1) the problem is not solvable for any $\lambda > 0$;

2) the problem is solvable for all $\lambda > 0$;

3) the problem is solvable not for all $\lambda > 0$.

Corollary 3 means that in the last case there exists $\lambda = \lambda_{\text{cr}} > 0$ such that for $\lambda < \lambda_{\text{cr}}$ the problem is solvable, while for $\lambda > \lambda_{\text{cr}}$ it is not solvable. From Corollary 2 it follows that, for $\sigma_1 \geqslant \sigma_2$, $\lambda_{\text{cr}}(\sigma_1) \geqslant \lambda_{\text{cr}}(\sigma_2)$. Let us consider these cases for the example of the problem

$$L(u) \equiv \sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \lambda a(x) F(u) = 0; \qquad \frac{du}{dT_x} + \sigma u = 0 \text{ or } u = 0 \text{ on } S; \tag{6}$$

$$a(x) > 0, \quad F(u) \text{ is defined for } u \geqslant 0 \text{ and } F(0) > 0.$$

Obviously, case 1) occurs when $\sigma(x) \equiv 0$ for any $F(u) > 0$. We shall assume that $\sigma(x) \not\equiv 0$.

Corollary 4. *Whatever the function $F(u)$ ($F(0) > 0$) may be, problem (6) is solvable for sufficiently small $\lambda > 0$.*

If $\lim\limits_{u\to\infty} F(u)/u = 0$, then problem (6) is solvable for any $\lambda > 0$.

Consider the problem

$$\sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + 1 = 0; \qquad \frac{du}{dT_x} = -k\sigma \text{ on } S. \tag{7}$$

Choose the constant $k > 0$ so that the solvability condition is satisfied (see (1)). Since the solution of problem (7) is determined up to a constant term, one may take a solution $u_0(x) \geqslant k > 0$. Then on S the condition $du_0/dT_x + \sigma u_0 \geqslant 0$ is fulfilled. Taking αu_0 with $\alpha > 0$, we obtain

$$L(\alpha u_0) = -\alpha + \lambda a(x) F(\alpha u_0).$$

Obviously, for $\alpha = 1$ and sufficiently small λ, $L(u_0) \leqslant 0$. If, however, $F(u)/u \to 0$ as $u \to \infty$, then $L(\alpha u_0) \leqslant 0$ for any $\lambda > 0$ for sufficiently large α.

For the Dirichlet problem the proof is analogous.

It is also obvious that if $F(u_1) = 0$ for some $u_1 > 0$, then problem (6) is solvable for any $\lambda > 0$. Excluding this case, which is trivial from the point of view of solvability, we shall assume that $F(u) > 0$ for $u > 0$.

Theorem 2*. *For problem (6) to be solvable for any $\lambda > 0$, it is necessary that $\varliminf\limits_{u\to\infty} F(u)/u = 0$, and it is sufficient that $\lim\limits_{u\to\infty} F(u)/u = 0$.*

In order that problem (6) be solvable not for all $\lambda > 0$, it is necessary and sufficient that $\min\limits_{u>0} F(u)/u$ be positive.

It is enough to prove the first assertion. Suppose that problem (6) is solvable for any $\lambda > 0$, and let λ_0 and $u_0(x)$ denote the least eigenvalue and the corresponding eigenfunction of the problem

$$\sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \mu a(x) u = 0; \qquad \frac{du}{dT_x} + \sigma u = 0 \text{ or } u = 0 \text{ on } S. \tag{8}$$

Multiplying equation (6) by $u_0(x)$ and integrating over the domain, we find

$$\frac{\lambda_0}{\lambda} = \int_G a(x) F(u) u_0 \, dx \Big/ \int_G a(x) u u_0 \, dx \geq \min_{u \leq M_\lambda} \frac{F(u)}{u}, \tag{9}$$

where $M_\lambda = \max u(x)$. Let u_λ denote a point of minimum in (9). As $\lambda \to \infty$, the set $\{u_\lambda\}$ cannot be bounded; otherwise, at a limiting point $\tilde{u}$ we would obtain $F(\tilde{u}) = 0$. Thus, for some sequence $\lambda_n \to \infty$, $u_{\lambda_n} \to \infty$ and $F(u_{\lambda_n})/u_{\lambda_n} \to 0$.

Sufficiency was already proved above.

It is interesting to note that from inequality (9) one obtains the following upper estimate for λ_{cr}:

$$\lambda_{\text{cr}} \leq \frac{\lambda_0}{\min\limits_{u>0} F(u)/u}. \tag{10}$$

In the theory of thermal explosion, i.e. for the equation $\Delta u + \lambda e^u = 0$, estimate (10) gives a good approximation to the exact value of λ_{cr}.

II. Let us consider the question of stability of solutions. We start from the following definition of stability.

Definition. A solution $u_0(x)$ of problem (1)–(2) [(1), (3)] will be called **stable** if, for sufficiently small in absolute value continuous $\psi(x)$, the solution $u(x,t)$ of the problem

$$\partial u/\partial t = L(u); \qquad u\big|_{t=0} = u_0 + \psi; \qquad du/dT_x + \sigma u = \varphi \ (u = f) \text{ on } S$$

tends, as $t \to \infty$, to $u_0(x)$ uniformly in x.

Put

$$L'(v) = \lim_{t\to 0} \frac{1}{t} L(u_0 + tv).$$

Theorem 3. *Let the boundary-value problem*

$$L'(v) + \mu v = 0; \qquad dv/dT_x + \sigma v = 0 \ (v = 0) \text{ on } S \tag{11}$$

be self-adjoint. Then, for stability of the solution $u_0(x)$ of problem (1)–(2) $[(1),(3)]$, it is necessary and sufficient that the spectrum of problem (11) be positive.

Theorem 4. *Let the solution $u_0(x)$ of problem (6) be stable. Then there exist positive functions $\underline{v}$ and $\overline{v}$ such that the inequalities*

$$L(u_0 + \overline{v}) \le -\varepsilon\overline{v} < 0 < \varepsilon\underline{v} \le L(u_0 - \underline{v}),$$

$$d\overline{v}/dT_x + \sigma\overline{v} \ge 0 \ge d\underline{v}/dT_x + \sigma\underline{v} \quad \text{or} \quad \underline{v} \le 0 \le \overline{v} \text{ on } S$$

are satisfied.

---

* In the case of the Dirichlet problem this theorem, as well as estimate (10), was obtained in [10].

for some $\varepsilon > 0$, and in the interval $u_0 - \bar{v} \leqslant u \leqslant u_0 + \bar{v}$ problem (6) has no other solutions.

This simple theorem means, in particular, that a stable solution is separated by means of upper and lower functions, i.e., functions satisfying condition A of Theorem 1.

Theorem 5. Let $\underline{u}(x) < \bar{u}(x)$ be lower and upper functions for problem (1), (2) or (1), (3). Suppose that in the interval $\underline{u}(x) \leqslant u \leqslant \bar{u}(x)$ there exists a nonnegative and nondecreasing (nonpositive and nonincreasing) derivative $\partial a/\partial u$ with respect to u, and, moreover,

$$L(\bar{u}) \leqslant -\varepsilon(\bar{u} - \underline{u}), \qquad L(\underline{u}) \geqslant \varepsilon(\bar{u} - \underline{u})$$

for some $\varepsilon > 0$. Then there exists a unique solution $u_0(x)$ satisfying the inequality

$$\underline{u}(x) < u_0(x) < \bar{u}(x),$$

and this solution is stable.

If $\tilde{u}$ is a minimal (maximal) solution, then it is easy to show that

$$L(\tilde{u} + \alpha\tilde{v}) \leqslant -a\varepsilon\tilde{v} \quad (L(\tilde{u} - \alpha\tilde{v}) \geqslant a\varepsilon\tilde{v})$$

for $0 \leqslant \alpha \leqslant 1$. Here $\tilde{v} = \bar{u} - \tilde{u}$ $(= \tilde{u} - \underline{u})$. From these inequalities, if one takes $\alpha = e^{-\varepsilon t}$, it follows that any solution coincides with the minimal (maximal) one, and also stability follows.

Corollary. Suppose that in problem (6) $F'(u)$ is a nondecreasing function.

1. From Theorems 4 and 5 it follows easily that the minimal solution is always stable, and only it is.
2. If the solution $\tilde{u}$ is unstable, then for no $\bar{u} \geqslant \tilde{u}$ and $\varepsilon > 0$ can the inequality $L(\bar{u}) \leqslant -\varepsilon\bar{u}$ be satisfied.

Theorem 6. Suppose that in problem (6) $F'(u)$ is an increasing positive function. Then, if problem (6) is solvable for $\lambda = \lambda_{\text{cr}}$, the solution $u_{\text{cr}}(x)$ is unique and λ_{cr} coincides with the first eigenvalue μ_0 of the problem

$$\sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right) + \mu a(x)F'(u_{\text{cr}})\,v = 0; \qquad \frac{dv}{dT_x} + \sigma v = 0 \text{ or } v = 0 \text{ on } S.$$

The assumption of nonuniqueness of u_{cr} leads to the fact that $\mu_0 > \lambda_{\text{cr}}$, and from the latter follows the solvability of problem (6) for $\lambda > \lambda_{\text{cr}}$. Taking into account that

$$u_{\text{cr}}(x) = \lim_{\lambda \to \lambda_{\text{cr}}} u_\lambda(x)$$

of stable solutions of problem (7) for $\lambda < \lambda_{\text{cr}}$, one easily obtains $\mu_0 \geqslant \lambda_{\text{cr}}$.

In conclusion the author expresses gratitude to A. I. Volpert for his attention to the work and valuable advice.

Institute of Chemical Physics
Academy of Sciences of the USSR

Received
6 IX 1963

REFERENCES

1. K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.
2. V. Pogozhel' skii, *Mat. sborn.*, 47 (89), No. 4, 397 (1959).
3. O. A. Oleinik, *Mat. sborn.*, 30, No. 3, 695 (1952).
4. A. Friedman, *J. Math. Mech.*, 7, No. 3, 393 (1958).
5. M. Nagumo, *Osaka Math. J.*, 6, 207 (1954).
6. I. M. Gel' fand, *UMN*, 14, issue 2, 87 (1959).
7. D. A. Frank-Kamenetskii, *Diffusion and Heat Transfer in Chemical Kinetics*, Moscow-Leningrad, 1947.
8. S. I. Pokhozhaev, *DAN*, 138, No. 2, 305 (1961).
9. K. Akô, *J. Math. Soc. Japan*, 13, No. 1, 45 (1961).
10. S. I. Khudyaev, *DAN*, 148, No. 1, 44 (1963).

Note: Figure translations are in progress. See original paper for figures.