MATHEMATICS

1964

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Abstract

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MATHEMATICS

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ON A GENERALIZATION OF THE TRICOMI PROBLEM

(Presented by Academician M. A. Lavrent' ev, 23 V 1964)

In the present note we consider a boundary-value problem for the Tricomi equation

$$yu_{xx} + u_{yy} = 0, \tag{1}$$

which is a generalization of the Tricomi problem (1) in the case when the unknown function $u(x, y)$ and its partial derivative $u_y(x, y)$ have discontinuities of the first kind on the line of parabolic degeneration.

Let D be a simply connected finite domain of the plane xOy, bounded by a Jordan curve σ with endpoints at the points $A(-1, 0)$, $B(1, 0)$, situated in the upper half-plane $y > 0$, and by the segments AC and BC of the characteristics $x + 1 = (2/3)(-y)^{3/2}$ and $1 - x = (2/3)(-y)^{3/2}$ of equation (1), issuing from the point $C[0, -(3/2)^{2/3}]$. Denote by D_1 and D_2, respectively, the elliptic and hyperbolic parts of the mixed domain D.

Problem T_α. It is required to determine a function $u(x, y)$ with the following properties: 1) $u(x, y)$ is a solution of equation (1) in the domain D for $y \neq 0$; 2) $u(x, y)$ is continuous in the closed domain $\overline{D}$ for $y \neq 0$; 3) the partial derivatives u_x and u_y are continuous in the domain D for $y \neq 0$, and at the points A and B they may tend to infinity of order lower than $2/3$; 4) on the line of parabolic degeneration the functions $u(x, y)$ and $u_y(x, y)$ satisfy the gluing conditions

$$u(x, -0) = \alpha(x)u(x, +0) + \gamma(x), \qquad -1 \leq x \leq 1,$$

$$u_y(x, -0) = \beta(x)u_y(x, +0) + \delta(x), \qquad -1 < x < 1, \tag{2}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $\delta(x)$ are given functions, with $\beta(x)$ and $\gamma(x)$ differentiable and $\beta'(x)$, $\gamma'(x)$ satisfying a Hölder condition for $-1 \leq x \leq 1$, while $\delta(x)$ and $\alpha(x)$ are, respectively, twice and three times continuously differentiable for $-1 \leq x \leq 1$; $\beta(x) \neq 0$ for $-1 \leq x \leq 1$ and

$$\alpha(x)\beta(x) \geq 0, \qquad \beta(x) \int_{-1}^{x} \frac{\alpha'(t)}{(x-t)^{1/3}}\, dt \geq 0, \qquad -1 < x < 1; \tag{3}$$

5) $u(x, y)$ assumes the prescribed values

$$u = \varphi \quad \text{on } \sigma, \tag{4}$$

$$u = \psi(x) \quad \text{on } AC, \tag{5}$$

where φ and ψ are twice continuously differentiable functions, $\psi''(x)$ satisfies a Hölder condition for $-1 \leq x \leq 0$, and $\varphi(A) = \alpha(A)\psi(A) + \gamma(A)$.

For $\alpha(x) = \beta(x) = 1$, $\gamma(x) = \delta(x) = 0$, the problem T_α coincides with the Tricomi problem.

In the domain D_2, by virtue of (2), the solution $u(x, y)$ has the form

$$u(x, y) = 2^{2/3} k_1 \int_{-1}^{1} [\alpha(v)\tau(v) + \gamma(v)](1-t^2)^{-5/6}\, dt +$$

$$+ \left(\frac{2}{3}\right)^{2/3} k_2 y \int_{-1}^{1} [\beta(v)\nu(v) + \delta(v)](1-t^2)^{-1/6}\, dt, \tag{6}$$

where $\tau(x) = u(x, +0)$, $\nu(x) = u_y(x, +0)$, $-1 \leq x \leq 1$, and $v = x + \frac{2}{3}(-y)^{3/2} t$, $k_1 = \Gamma(1/3)/\Gamma^2(1/6)$, $k_2 = (3/4)^{2/3}\Gamma(5/3)/\Gamma^2(5/6)$.

By virtue of (5), from (6) we obtain

$$\alpha(x)\tau(x) + \gamma(x) = \psi_1(x) + k \int_{-1}^{x} \frac{\beta(t)\nu(t) + \delta(t)}{(x-t)^{1/3}}\, dt, \tag{7}$$

where $k = 3^{2/3}\Gamma^3(1/3)/4\pi^2$ and

$$\psi_1(x) = \frac{(x+1)^{5/6}}{2k_1\pi} \frac{d}{dx} \int_{-1}^{1} \frac{\psi\left(\frac{t-1}{2}\right)}{(x-t)^{1/6}(t+1)^{2/3}}\, dt.$$

For $\psi(x) = \gamma(x) = \delta(x) \equiv 0$, from (7), applying Abel' s inversion formula, we obtain

$$\beta(x)\nu(x) = \frac{\sqrt{3}}{2k\pi} \frac{d}{dx} \int_{-1}^{x} \frac{\alpha(t)\tau(t)}{(x-t)^{2/3}}\, dt.$$

Hence, taking (3) into account, just as in the Tricomi problem ([2]), we conclude that the solution $u(x, y)$ attains a positive maximum and a negative minimum in the closed domain $\overline{D}_1$ on the curve σ (the extremum principle). From this principle the uniqueness of the solution of problem T_α follows immediately.

The existence of a solution of problem T_α will be proved if the function $\nu(x)$ can be determined (1). We shall assume that σ coincides with the normal contour $\sigma_0 : x^2 + \frac{4}{9}y^3 = 1$ and that $\varphi(A) = \psi(A) = \varphi(B) = 0$, $\varphi'(A) = \psi'(A) = \varphi'(B) = 0$, where the derivatives are taken in the direction tangent to the contour $\sigma_0 + AC$.

The relation between $\tau(x)$ and $\nu(x)$, brought from the elliptic part of the domain D, has the form

$$\tau(x) + k\int_{-1}^{1}\left[|t-x|^{-1/3} - (1-tx)^{-1/3}\right]\nu(t)\,dt = F^*(x), \tag{8}$$

where

$$F^*(x) = \frac{k}{2^{1/3}3^{7/3}}(1-x^2)\int_{-1}^{1}(1-2tx+x^2)^{-7/6}(1-t^2)^{-1/3}\varphi(t)\,dt.$$

We shall seek the function $\nu(x)$ in the class H^* on the segment $[-1, 1]$ (the definition of the class H^* is given in ([4])). Let $-1 < x < 1$. Eliminating $\tau(x)$ from (7) and (8), to determine $\nu(x)$ we obtain the singular integral equation

$$[\alpha(x) + 2\beta(x)]\nu(x) + \frac{\sqrt{3}\,\alpha(-1)}{\pi}\int_{-1}^{1}\left(\frac{t+1}{x+1}\right)^{2/3}\left(\frac{1}{t-x} - \frac{1}{1-tx}\right)\nu(t)\,dt +$$

$$+\int_{-1}^{1}\frac{k_1(x,t)\nu(t)}{t-x}\,dt + \int_{-1}^{1}\frac{k_2(x,t)\nu(t)}{1-tx}\,dt + \int_{-1}^{1}k_3(x,t)\nu(t)\,dt = F(x), \tag{9}$$

where

$$f(x) = \frac{\sqrt{3}}{\pi}\frac{d}{dx}\int_{-1}^{x}\frac{\varphi^*(t)\,dt}{(x-t)^{2/3}},$$

$$\varphi^*(x) = \frac{1}{k}\left[\alpha(x)F^*(x) - \psi_1(x) + \gamma(x)\right] - \int_{-1}^{x}\frac{\delta(t)\,dt}{(x-t)^{1/3}},$$

$$k_1(x,t) = \frac{\sqrt{3}}{\pi}\int_{-1}^{x}\left(\frac{t-\xi}{x-\xi}\right)^{2/3}\alpha'(\xi)\,d\xi, \qquad k_2(x,t) = -\frac{\sqrt{3}}{\pi}\int_{-1}^{x}\left(\frac{1-t\xi}{x-\xi}\right)^{2/3}\alpha'(\xi)\,d\xi,$$

$$k_3(x,t) = \frac{2\sqrt{3}\,\omega(x,t)}{\pi}\frac{1}{x-t}\int_t^x \left(\frac{t-\xi}{x-\xi}\right)^{2/3} \alpha'(\xi)\,d\xi,$$

and $\omega(x,t) = 1$ for $t \in [-1,x]$, $\omega(x,t) = 0$ for $t \bar{\in} [-1,x]$.

In addition, we shall assume that

$$\alpha(-1) = \alpha'(1) = 0. \tag{10}$$

In this case equation (9) takes the form

$$a(x)v(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\left(\frac{1}{t-x} - \frac{1}{1-tx}\right)v(t)\,dt + \int_{-1}^{1} k(x,t)v(t)\,dt = F(x), \tag{11}$$

where $a(x) = \alpha(x) + 2\beta(x)$, $b(x) = \sqrt{3}\alpha(x)$, and

$$k(x,t) = \frac{2}{\pi\sqrt{3}}\int_{-1}^{x}\frac{\alpha'(\xi)\,d\xi}{(x-\xi)^{2/3}[x+\theta(t-x)-\xi]^{1/3}}$$

$$-\frac{\sqrt{3}}{\pi}\frac{1}{1-tx}\int_{-1}^{x}\left[\left(\frac{1-t\xi}{x-\xi}\right)^{2/3} - 1\right]\alpha'(\xi)\,d\xi + \frac{2\sqrt{3}\,\omega(x,t)}{\pi}\frac{1}{x-t}\int_t^x\left(\frac{t-\xi}{x-\xi}\right)^{2/3}\alpha'(\xi)\,d\xi,$$

$$0 < \theta < 1.$$

To equation (11) one can apply the well-known method of regularization ([2]). For this purpose we rewrite it in the form

$$a(x)v(x)\frac{b(x)}{\pi}\int_{-1}^{1}\left(\frac{1}{1-x} - \frac{1}{1-tx}\right)v(t)\,dt = g(x), \tag{12}$$

$$g(x) = F(x) - \int_{-1}^{1} k(x,t)v(t)\,dt. \tag{13}$$

It is not difficult to see that $g'(x)$ satisfies a Hölder condition for $-1 < x < 1$.

The solution of equation (12) is given by the formula ([3])

$$v(x) = A(x)g(x) - \frac{B(x)Z(x)}{\pi}\int_{-1}^{1}\frac{g(t)}{Z(t)}\left(\frac{1}{t-x} - \frac{1}{1-tx}\right)dt, \tag{14}$$

$$A(x) = \frac{a(x)}{a^2(x)+b^2(x)}, \qquad B(x) = \frac{b(x)}{a^2(x)+b^2(x)}, \qquad Z(x) = \sqrt{a^2(x)+b^2(x)}\, e^{\Gamma^*(x)}.$$

$$\Gamma^*(x) = \int_{-1}^{1} \theta(t) \left[\frac{1}{t-x} - \frac{1}{t(1-tx)}\right] dt, \qquad \theta(x) = -\frac{1}{\pi} \underset{(-\pi/2,\ \pi/2)}{\operatorname{arctg}} \frac{b(x)}{a(x)}.$$

Taking (13) into account, for $\nu(x)$ we obtain the Fredholm integral equation equivalent to equation (11)

$$\nu(x) + \int_{-1}^{1} H(x,t)\, \nu(t)\, dt = h(x), \tag{15}$$

where

$$H(x,t) = -A(x)k(x,t) = \frac{B(x)Z(x)}{\pi} \int_{-1}^{1} \frac{k(t_1,t)}{Z(t_1)} \left(\frac{1}{t_1-x} - \frac{1}{1-t_1x}\right) dt_1,$$

$$h(x) = A(x)F(x) - \frac{B(x)Z(x)}{\pi} \int_{-1}^{1} \frac{F(t)}{Z(t)} \left(\frac{1}{t-x} - \frac{1}{1-tx}\right) dt.$$

The kernel $H(x,t)$ has a moving singularity of order $1/3$ for $t = x$ and a fixed singularity of order less than $2/3$ for $x = 1$. Consequently, the Fredholm theorems are applicable to equation (15). From the uniqueness of the solution of problem T_α it follows that equation (15) is solvable. From (14) it follows directly that the solution $\nu(x)$ belongs to the class H^* on the interval $[-1,1]$ and is differentiable in the interval $(-1,1)$.

Remark 1. If the function $\alpha(x)$ is linear, i.e. $\alpha(x) = px + q$, the existence of a solution of the problem T_α can be proved without additional assumptions of the type (10). In the case where $\alpha(x) = C = \text{const}$, the solution $\nu(x)$ of equation (9) is found explicitly:

$$\nu(x) = A(x)F(x) - \frac{B(x)Z(x)}{\pi} \int_{-1}^{1} \frac{F(t)}{Z(t)} \left(\frac{t+1}{x+1}\right)^{2/3} \left(\frac{1}{t-x} - \frac{1}{1-tx}\right) dt.$$

From this formula, for $\alpha(x) = \beta(x) = 1$, $\gamma(x) = \delta(x) = 0$, one obtains the function $\nu(x)$ for the Tricomi problem.

Remark 2. The case where the curve σ has continuous curvature and ends in arcs AA' and BB' of an arbitrarily small length of a normal curve is investigated in the same way as in the Tricomi problem ([1,2]).

The author expresses gratitude to A. V. Bitsadze for helpful suggestions and attention to this work.

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Received
21 V 1964

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Note: Figure translations are in progress. See original paper for figures.