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Abstract

Full Text

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ON THE RELATION BETWEEN FINITELY PROJECTIVE AND FINITELY REDUCTION CLASSES OF MODELS

(Presented by Academician A. I. Mal' tsev, 30 XII 1963)

In 1958 A. I. Mal' tsev introduced an important concept for algebraic applications: that of a finitely reduction class. It follows directly from the definition of a finitely reduction class that every finitely projective class is finitely reduction.* The question whether the converse is true has until now remained open. A partial solution of it is given by

Mal' tsev' s Theorem. *Every finitely reduction class that contains no finite models is finitely projective.*

The proof of this theorem, contained in (^{1}), reveals more; namely, the following is true:

Theorem 1. *For every finitely reduction class K there exists a finitely projective class L such that $L \subset K$ and $K_{\aleph_0} \subset L$, where $K_{\aleph_0}$ is the class consisting of the infinite models in K.*

By $K^{\aleph_0}$ we shall denote the class $K \setminus K_{\aleph_0}$.

Theorem 1 shows that a solution of the question mentioned above may be expected by studying the properties of classes $K^{\aleph_0}$, where K are finitely reduction classes, and of classes $L^{\aleph_0}$, where L are finitely projective classes.

By the **finite spectrum** of a class of models we shall mean the set of cardinalities of its finite models.

It is obvious that

Theorem 2. *Whatever degree of axiom we take, the finite spectrum of the class defined by it is recursive.*

In particular, the finite spectrum of every finitely projective class is recursive.

We shall now show that there exists a finitely reduction class whose finite spectrum is not recursive. Moreover, we shall show that the following is true:

Theorem 3. *For every recursively enumerable set M not containing zero, there exists a finitely reduction class with finite spectrum M.*

The case of finite M is trivial. Let M be infinite and let Θ be a primitive recursive function with set of values M, and let m be the least value of Θ.

Let Σ^* be a system of equalities defining the function Θ as obtained from the functions $O(n) = 0$, $s(n) = n + 1$, $E^k_{i_l j_l}, \dots, E^k_{i_k j_k}$, where

$$E^k_{i_l j_l}(n_1, \dots, n_{j_l}, \dots, n_{i_l}) = n_{j_l} \quad (1 \le l \le k),$$

by means of substitutions and primitive recursions.

Consider an elementary formal theory with equality and nonlogical constants $\le, T$ and R, where $\le$ is a two-place predicate, T is a functional predicate of the same valency as the function Θ, and R is a one-place predicate. By definition,

$$x < y \leftrightarrow x \le y \wedge x \ne y.$$

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* For definitions of the concepts from model theory used by us, see (1).

We specify the following system of axioms:

$$\begin{gathered} x \leqslant y \wedge y \leqslant x \to x = y, \\ x \leqslant y \wedge y \leqslant z \to x \leqslant z, \\ x \leqslant y \vee y \leqslant x, \\ (\exists x)(y)[x \leqslant y], \\ (x)(\exists y)(z)[x < y \wedge (x < z \to y \leqslant z)]. \end{gathered}$$

We shall denote this system of axioms by Σ_1. It defines the class of linearly ordered sets with least elements such that every element has an immediate successor.

By 0 we shall denote that x for which $(y)[x \leqslant y]$. By definition, $y = S(x)$ if and only if

$$x < y \wedge (z)[x < z \to y \leqslant z].$$

Let $\mathfrak{M}$ be any model satisfying Σ_1. Those of its elements whose sets of minorants are finite will be called **natural numbers**. The submodel consisting of all natural numbers will be called the Z-submodel. It is obvious that the $\langle 0, S \rangle$-projection of the Z-submodel is a model of Peano arithmetic.

We define the functional predicates $\mathfrak{D}, E_{i_1 j_1}, \dots, E_{i_k j_k}$ as follows:

$$\mathfrak{D}(x) = 0;$$

$$E_{i_l j_l}(x_1, \dots, x_{j_l}, \dots, x_{i_l}) = x_{j_l} \quad (1 \leqslant l \leqslant k).$$

Replacing everywhere in the system Σ^* the symbols $O, S, E_{i_1 j_1}, \dots, E_{i_k j_k}, \Theta$ respectively by the symbols $\mathfrak{D}, S, E_{i_1 j_1}, \dots, E_{i_k j_k}, T$, and the number variables by individual variables (with the corresponding indices) of our theory, we obtain a system of axioms Σ_2.

It is obvious that, under the conditions Σ_2, the function T is uniquely defined on the natural numbers, or, more precisely, the Z-submodels of any models satisfying the system of axioms $\Sigma_1 \cup \Sigma_2$ are isomorphic.

By x_m we shall denote that x for which

$$(\exists y_0) \dots (\exists y_{m-2})\{y_0 \neq y_1 \wedge \dots \wedge y_{m-3} \neq y_{m-2} \wedge y_0 < x \wedge \dots \wedge y_{m-2} < x \wedge$$

$$\wedge (z)[z < x \to (z = y_0 \vee \dots \vee z = y_{m-2})]\}, \quad \text{if } m > 1.$$

In the case $m = 1$ we put $x_m = 0$.

By Σ_3 we denote the following system of axioms:

1. $y < x \wedge R(x) \to R(y)$.
2. $(\exists x)(y)[R(x) \wedge (R(y) \to y \leqslant x)]$, i.e., there exists a greatest x for which $R(x)$ is true. We shall denote such an x by x_R.
3. $$x_R = x_m \vee (\exists x)(\exists y)\{y = T(x) \wedge$$ $$\wedge (\exists z)(u)[S(x_R) = T(z) \wedge y < T(z) \wedge (y < T(u) \to z \leqslant u)]\}.$$

We shall denote the system of axioms $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ by Σ.

Let Σ define a class of models K, whose R-reduction is the class L. We shall show that the finite spectrum of L is the set M.

Let $n \in M$, and let $\mathfrak{M}$ be a model satisfying Σ, coinciding with its Z-submodel and such that the predicate R is true on its first n elements and only on them. It is easy to verify that $\mathfrak{M} \in K$. The R-reduction of $\mathfrak{M}$ is an n-element model. Thus every $n \in M$ belongs to the finite spectrum of the class L.

It remains to show that every number from the finite spectrum of L belongs to M.

Let $\mathfrak{M}$ be an arbitrary model from K, and let x_R be a natural number (otherwise the R-reduction of the model $\mathfrak{M}$ is infinite). If $x_R = x_m$, then

The **R**-reduction $\mathfrak{M}$ is an m-element submodel. But m belongs to M. Suppose x_R does not coincide with x_m. Then, according to axiom 3 from Σ_3, there will be x and y such that $y = \mathbf{T}(x) \wedge \mathbf{S}(x_R) = \mathbf{T}(z)$, where z is the least of those u for which $y < \mathbf{T}(u)$. It is clear that y is a natural number (as the minorant of the natural number x_R). Since the function Θ, by hypothesis, is unbounded on the set of natural numbers, there will be a natural number u such that $y < \Theta(u)$. Consequently, z, the least of such u, will also be a natural number. Since the function $\mathbf{T}$ is uniquely defined on the set of natural numbers, the minorant number of the element x_R belongs to M. The theorem is proved.

From Theorems 2 and 3 it follows

Theorem 4. *There exist finitely reductional classes which are not finitely projective.*

Moreover, there exists a finitely reductional class not definable by any axiom (of whatever degree). Such is any finitely reductional class with a nonrecursive finite spectrum.

At the same time, using Theorem 1, it is easy to show that every finitely reductional class is defined by a system (in general infinite) of second-order axioms.

In connection with Theorems 1 and 4, the question arises whether there exists a criterion which, from the properties of the class K^{κ_0}, permits one to determine whether the finitely reductional class K is finitely projective. This question will be considered elsewhere.

From Theorem 1 and from the fact that every finite semigroup embeddable in a group is a group, it follows that the class of semigroups embeddable in groups is finitely projective.

The proof of A. I. Mal' cev' s theorem contains a method which makes it possible to find an axiom defining the class of such semigroups. The class of semigroups embeddable in generalized groups is also finitely projective (3).

It is known that for every finitely axiomatizable class of models K, the class of all submodels of models from K is finitely reductional. Is every such class finitely projective? This question remains open.

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CITED LITERATURE

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[3] B. M. Schein, *DAN*, **134**, No. 5 (1960).

Note: Figure translations are in progress. See original paper for figures.

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