

GROWTH OF A MEROMORPHIC FUNCTION ALONG A RAY

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1964

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Abstract

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MATHEMATICS

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GROWTH OF A MEROMORPHIC FUNCTION ALONG A RAY

(Presented by Academician M. A. Lavrent'ev, November 25, 1963)

§ 1. In Nevanlinna theory the growth of entire functions is characterized by the quantity

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta$$

and the order (lower order) of growth is called

$$\overline{\lim}_{r \rightarrow \infty} [\ln r]^{-1} \ln T(r) \quad \left(\underline{\lim}_{r \rightarrow \infty} [\ln r]^{-1} \ln T(r) \right). \quad (1,1)$$

In the classical theory of entire functions, for this purpose the quantity

$$M(r) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$$

is used, and the order (lower order) is called

$$\overline{\lim}_{r \rightarrow \infty} [\ln r]^{-1} \ln \ln M(r) \quad \left(\underline{\lim}_{r \rightarrow \infty} [\ln r]^{-1} \ln \ln M(r) \right). \quad (1,2)$$

The relation between the quantities $T(r)$ and $\ln M(r)$ was investigated in works ^(1-3,5,8) and at present has not been fully studied.

R. Nevanlinna ⁽¹⁾ proved the inequality

$$T(r) \leq \ln M(r) \leq \frac{k+1}{k-1} T(kr) \quad (k > 1),$$

from which it follows that both definitions (1,1), (1,2) of order and lower order are equivalent. (We agree to denote by ρ the order of a meromorphic function, and by λ its lower order.) Paley ⁽²⁾ proved that for any $0 \leq \rho \leq \infty$ there exists an entire function of order ρ for which the equality

$$\overline{\lim}_{r \rightarrow \infty} [T(r)]^{-1} \ln M(r) = +\infty$$

holds, and put forward the hypothesis: for an entire function of order ρ the inequality

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{T(r)} \leq \begin{cases} \pi\rho \operatorname{csc} \pi\rho, & 0 \leq \rho \leq 0.5, \\ \pi\rho, & \rho > 0.5. \end{cases} \quad (1,3-1,4)$$

This hypothesis was proved by Valiron ⁽³⁾ only for $0 \leq \rho \leq 0.5$. For $\rho > 0.5$ the question remains open. A. A. Gol'dberg ⁽⁶⁾ obtained the following result:

$$\overline{\lim}_{r \rightarrow \infty} [T(r)]^{-1} \ln^+ |f(re^{i\varphi})| \leq \pi\rho \quad (\rho > 0.5, 0 \leq \varphi < 2\pi). \quad (1,5)$$

From this result follows the validity of Paley's hypothesis for functions satisfying the condition $|f(re^{i\varphi})| \sim M(r)$ for some fixed φ .

A. A. Gol'dberg and I. V. Ostrovskii ^(4,7) strengthened Valiron's result by proving that, for entire functions of lower order $\lambda \leq 0.5$, in inequality (1.3) one may replace ρ by λ .

In the present paper we prove that, for an entire function of lower order $\lambda > 0.5$, in inequality (1.5) one may replace ρ by λ . This result is a special case of a more general result obtained in this paper*:

Theorem. Let $f(z)$ be a meromorphic function of finite lower order $\lambda > 0.5$. The inequality

$$\lim_{r \rightarrow \infty} [T(r)]^{-1} \ln^+ |f(re^{i\varphi})| \leq \pi\lambda$$

holds.

§ 2. We shall carry out the proof of the theorem for the case $\lambda < \rho$. First we establish several auxiliary relations. Let

$$G_{\alpha,R} = \{z : 0 < |z| < R, |\arg z| < \alpha\},$$

where $0 < \alpha < \pi$, $R > 0$; let $g_{\alpha,R}(z, \zeta)$ be the Green function for $G_{\alpha,R}$.

Lemma. For a function meromorphic in $G_{\alpha,R}$ the following inequality holds

$$\begin{aligned} \ln^+ |f(r)| \leq & \frac{1}{2\pi} \int_0^R \{\ln^+ |f(te^{i\alpha})| + \ln^+ |f(te^{-i\alpha})|\} P(t, r, \alpha) dt + \\ & + \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \ln^+ |f(Re^{i\theta})| P(R, r, \alpha, \theta) d\theta + \sum_{b_k \in G_{\alpha,R}} \ln \left| \frac{r^\sigma + \overline{b_k^\sigma}}{r^\sigma - b_k^\sigma} \right|, \end{aligned} \quad (2.1)$$

where $\sigma = 0.5 \pi \alpha^{-1}$,

$$P(t, r, \alpha) = \pi \alpha^{-1} t^{\sigma-1} r^{\sigma} (t^{2\sigma} + r^{2\sigma})^{-1},$$

$$P(R, r, \alpha, \theta) = \sigma (R^{2\sigma} - r^{2\sigma}) \frac{4R^{\sigma} r^{\sigma} \cos \sigma \theta}{R^{4\sigma} - 2R^{2\sigma} r^{2\sigma} \cos 2\sigma \theta + r^{4\sigma}},$$

and b_k are the poles of $f(z)$.

This lemma follows from the known representation for $\ln |f(z)|$ in the sector $G_{\alpha, R}$ and from the following relations:

$$\left| \frac{\partial}{t \partial \varphi} g_{\alpha, R}(te^{i\varphi}, r) \right|_{\varphi=\pm\alpha} \leq P(t, r, \alpha),$$

$$\frac{\partial}{\partial R} g_{\alpha, R}(Re^{i\theta}, r) = R^{-1} P(R, r, \alpha, \theta),$$

$$g_{\alpha, R}(r, a) \leq \ln |(r^{\sigma} + \bar{a}^{\sigma})(r^{\sigma} - a^{\sigma})^{-1}|.$$

We now proceed to the proof of the theorem. Without loss of generality, we shall assume $\varphi = 0$, $|f(0)| < 1$. Choose γ so that $\lambda < \gamma < \rho$, and take $\alpha < \pi(2\gamma)^{-1}$; then from inequality (2.1) we find:

$$\begin{aligned} \int_{r_0}^{0.5R} r^{-\gamma-1} \ln^+ |f(r)| dr &\leq \frac{1}{2\pi} \int_0^R \int_{r_0}^{0.5R} \{\ln^+ |f(te^{i\alpha})| + \ln^+ |f(te^{-i\alpha})|\} \times \\ &\times r^{-\gamma-1} P(t, r, \alpha) dt dr + \frac{1}{2\pi} \int_{r_0}^{0.5R} \int_{-\alpha}^{\alpha} \ln^+ |f(Re^{i\theta})| r^{-\gamma-1} P(R, r, \alpha, \theta) dr d\theta + \\ &+ \sum_{b_k \in G_{\alpha, R}} \int_{r_0}^{0.5R} r^{-\gamma-1} \ln |(r^{\sigma} + \bar{b}_k^{\sigma})(r^{\sigma} - b_k^{\sigma})^{-1}| dr \end{aligned} \quad (2.2)$$

where $0 < r_0 < 0.5R$.

* We use the standard notation of Nevanlinna theory.

Noting that*

$$\begin{aligned} \int_{r_0}^{0.5R} r^{-\gamma-1} P(t, r, \alpha) dr &= t^{-\gamma-1} \int_{r_0 t^{-1}}^{0.5R t^{-1}} s^{-\gamma-1} P(1, s, \alpha) ds \leq \\ &\leq t^{-\gamma-1} \int_0^{\infty} s^{-\gamma-1} P(1, s, \alpha) ds = \pi t^{-\gamma-1} \sec \alpha \gamma, \\ \int_{r_0}^{0.5R} r^{-\gamma-1} \ln \left| \frac{r^{\sigma} + \bar{b}_k^{\sigma}}{r^{\sigma} - b_k^{\sigma}} \right| dr &\leq \int_{r_0}^{0.5R} r^{-\gamma-1} \ln \left| \frac{r^{\sigma} + |b_k|^{\sigma}}{r^{\sigma} - |b_k|^{\sigma}} \right| dr = \end{aligned}$$

$$\begin{aligned}
 &= |b_k|^{-\gamma} \int_{r_0|b_k|^{-1}}^{0.5R|b_k|^{-1}} s^{-\gamma-1} \ln |(s^\sigma + 1)(s^\sigma - 1)^{-1}| ds \leq \\
 &\leq |b_k|^{-\gamma} \int_0^\infty s^{-\gamma-1} \{\ln(s^\sigma + 1) - \ln|s^\sigma - 1|\} ds = \pi\gamma^{-1}|b_k|^{-\gamma} \tan \alpha\gamma,
 \end{aligned}$$

$$P(R, r, \alpha, \theta) \leq C_1 \alpha^{-1} r^\sigma R^{-\sigma} \quad \text{for } 0 < r < 0.5R,$$

from (2.2) we find

$$\begin{aligned}
 2 \cos \alpha\gamma \int_{r_0}^{0.5R} r^{-\gamma-1} \ln^+ |f(r)| dr &\leq \int_{r_0}^{0.5R} r^{-\gamma-1} \{\ln^+ |f(re^{i\alpha})| + \ln^+ |f(re^{-i\alpha})|\} dr + \\
 &+ 2\pi\gamma^{-1} \sin \alpha\gamma \sum_{r_0 < |b_k| < 0.5R} |b_k|^{-\gamma} + C_2 \sum_{|b_k| \leq r_0} |b_k|^{-\gamma} + \\
 &+ C_3 \sum_{0.5R < |b_k| < R} |b_k|^{-\gamma} + r_0 \int_0^{r_0} r^{-\gamma-1} \{\ln^+ |f(re^{i\alpha})| + \ln^+ |f(re^{-i\alpha})|\} dr + \\
 &+ C_4 T(R) R^{-\gamma} + \int_{0.5R}^R r^{-\gamma-1} \{\ln^+ |f(re^{i\alpha})| + \ln^+ |f(re^{-i\alpha})|\} dr.
 \end{aligned}$$

Integrating the resulting inequality with respect to α over the interval from 0 to $\pi(2\gamma)^{-1}$, we find

$$\begin{aligned}
 (\pi\gamma)^{-1} \int_{r_0}^{0.5R} r^{-\gamma-1} \ln^+ |f(r)| dr &\leq \int_{r_0}^{0.5R} r^{-\gamma-1} m(r, \infty) dr + \\
 &+ \gamma^{-2} \int_{r_0}^{0.5R} r^{-\gamma} dn(r, \infty) + C_5 T(2R) R^{-\gamma} + C_6 = \\
 &= \int_{r_0}^{0.5R} r^{-\gamma-1} m(r, \infty) dr + \int_{r_0}^{0.5R} r^{-\gamma-1} N(r, \infty) dr + C_7 T(2R) R^{-\gamma} + C_8. \quad (2.3)
 \end{aligned}$$

Since $\gamma < \rho$, as $R \rightarrow \infty$,

$$\int_{r_0}^{0.5R} r^{-\gamma-1} T(r) dr \rightarrow \infty,$$

* Throughout the article, by the letter C with subscripts we shall denote positive constants independent of R , r , and α .

on the other hand, since $\gamma > \lambda$, we have

$$\lim_{R \rightarrow \infty} R^{-\gamma} T(R) = 0.$$

Therefore from (2.3) we find

$$\lim_{R \rightarrow \infty} \left(\int_{r_0}^{0.5R} r^{-\gamma-1} \ln^+ |f(r)| dr \right) \left\{ \int_{r_0}^{0.5R} r^{-\gamma-1} T(r) dr \right\}^{-1} \leq \pi\gamma,$$

whence

$$\lim_{r \rightarrow \infty} [T(r)]^{-1} \ln^+ |f(r)| \leq \pi\gamma.$$

Letting now γ tend to λ , we obtain the assertion of the theorem for the case $\lambda < \rho$.

For $\lambda = \rho$ one should apply the method of work ⁶.

Remark. From the results of A. A. Gol'dberg and I. V. Ostrovskii ^{4,7} there follows the following assertion, generalizing inequalities (1, 3) to meromorphic functions:

Let $f(z)$ be a meromorphic function of finite lower order $\lambda \leq 0.5$. The inequality holds

$$\lim_{r \rightarrow \infty} [T(r)]^{-1} \ln M(r) \leq \pi\lambda \csc \pi\lambda (1 - \cos \pi\lambda(1 - \Delta(\infty))),$$

where

$$\Delta(\infty) = 1 - \lim_{r \rightarrow \infty} N(r, \infty)[T(r)]^{-1}.$$

I express my deep gratitude to I. V. Ostrovskii for supervising the work.

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Received
20 XI 1963

REFERENCES

- ¹ R. Nevanlinna, *Single-Valued Analytic Functions*, Moscow-Leningrad, 1941.
- ² R. E. A. C. Paley, Proc. Cambridge Phil. Soc., 28, 262 (1932).
- ³ G. Valiron, *Mathematica*, 11, 264 (1935).
- ⁴ A. A. Gol' dberg, I. V. Ostrovskii, Notes of the Mathematical Department of Kharkov State University and the Kharkov Mathematical Society, 27, ser. 4, 3 (1961).
- ⁵ A. A. Gol' dberg, Dokl. AN URSS, 4, 443 (1963).
- ⁶ A. A. Gol' dberg, DAN, 152, No. 5 (1963).
- ⁷ I. V. Ostrovskii, DAN, 150, 32 (1963).
- ⁸ I. V. Ostrovskii, Notes of the Mathematical-Mechanical Faculty of Kharkov State University and the Kharkov Mathematical Society, 28, ser. 4, 23 (1961).

Note: Figure translations are in progress. See original paper for figures.

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