ON A NONSTATIONARY MAGNETOGASDYNAMIC PROBLEM. AN ANALOGUE OF THE RIEMANN WAVE

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Abstract

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We consider a nonstationary problem of gas dynamics and magnetic gas dynamics in the presence of two cyclic coordinates. Finding a complete set of symmetry integrals, we use them to transform the original equations to a system of two scalar equations for determining the total pressure and the first component of the velocity. Passing in this system to the variables (P, ψ, Θ), we obtain an equation which, up to notation, coincides with the equation of L. I. Sedov in the form of Yu. V. Rudnev (^{1,2}). Using certain techniques developed in the analysis of stationary gasdynamic problems, we arrive at analogous results in the nonstationary problem; Riemann waves in a "quasi-barotropic" medium are studied. This approach is applicable, in particular, also to the analysis of various nonrelativistic variants of magnetic gas dynamics (^{3}).

1. Initial assumptions. Symmetry integrals

The system of equations of relativistic magnetic hydrodynamics consists of:

the Maxwell and continuity equations ($^{4-6}$):

$$\frac{\partial \mathbf{H}}{\partial \tau} = \operatorname{rot} \Omega \mathbf{U} \times \mathbf{H}, \qquad \operatorname{div} \mathbf{H} = 0, \qquad \frac{\partial}{\partial \tau} \frac{n}{\Omega} + \operatorname{div} n\mathbf{U} = 0, \tag{1}$$

$$\mathbf{E} = -\Omega \mathbf{U} \times \mathbf{H}, \qquad \Omega = (1 + \mathbf{U}^2)^{-1/2}, \qquad d\tau = c\,dt;$$

the equations of motion, which in flux form have the form:

$$\frac{\partial}{\partial x_\beta} \left[c^2 n M \Omega U_\alpha U_\beta - \frac{1}{4\pi}(H_\alpha H_\beta + E_\alpha E_\beta) + \delta_{\alpha\beta} \left(p + \frac{\mathbf{H}^2 + \mathbf{E}^2}{8\pi} \right) \right] +$$

$$+ \frac{\partial}{\partial \tau} \left[c^2 n M U_\alpha + \frac{(\mathbf{E} \times \mathbf{H})_\alpha}{4\pi} \right] = 0, \qquad M = \frac{\varepsilon + p}{c^2 n \Omega} \quad (\alpha, \beta = 1, 2, 3); \tag{2}$$

the equation relating the thermodynamic quantities:

$$n^2 \frac{d}{d\tau}\frac{\varepsilon}{n} = p\frac{dn}{d\tau}, \qquad \frac{d}{d\tau} \equiv \frac{\partial}{\partial\tau} + \Omega(\mathbf{U}\nabla). \tag{3}$$

The three-dimensional vector $\mathbf{U}$ is made up of the spatial components of the four-dimensional velocity vector; n is the number of particles per unit proper volume; ε is the proper internal energy per unit proper volume. The equations are written in a three-dimensional form, which here turns out to be preferable (^{7}). Equation (3) is a consequence of equations (1), (2) and of the time component of the divergence of the energy-momentum tensor. In ordinary relativistic gas dynamics the latter circumstance was noted by V. S. Tkalich (^{7}) and I. S. Shikin (^{8}). The system (1)–(3) is supplemented by the thermodynamic identity ($^{4-6}$)

$$d\frac{\varepsilon + p}{n} = T\,d\frac{s}{n} + \frac{1}{n}\,dp. \tag{4}$$

Here s is the entropy per unit proper volume.

Let the coordinates (x_2, x_3) be cyclic $(\partial/\partial x_2 = \partial/\partial x_3 = H_1 = 0)$. Solving the continuity equation (1), we introduce the particle function ψ

$$\frac{\partial\psi}{\partial x} = \frac{n}{\Omega}, \qquad \frac{\partial\psi}{\partial t} = -\frac{nv}{\Omega}, \qquad v \equiv \Omega U_1, \quad x \equiv x_1. \tag{5}$$

Then from relations (3)–(5) we obtain

$$p = n^2\left(\frac{\varepsilon}{n}\right)_n, \qquad \varepsilon = \varepsilon(n, \psi), \qquad s = nf(\psi). \tag{6}$$

The quantity f depends arbitrarily on the particle function ψ. The subscript n denotes differentiation with respect to n at fixed ψ. We note that both the results presented below and the results of the authors' work (^{3}) are also valid for some other thermodynamic relations. For this it is necessary that the pressure p have the form $p = p(n, \psi)$.

Substituting (5) into the induction equation (1), and integrating it, we find the relation of the magnetic field to the density, the particle function, and the velocity. With the aid of the relation obtained, and also (5) and (6), we find the general solution of the second and third components of the equation of motion (2). As a result we obtain

$$\mathbf{H} = \sqrt{4\pi}\,\frac{n\sqrt{1+\mathbf{u}^2}}{\sqrt{1-v^2}}\,\mathbf{B}, \qquad \mathbf{u} = \frac{\varepsilon_n\vec{\Pi} + n(\vec{\Pi}\mathbf{B})\mathbf{B}}{\varepsilon_n^2 + n\varepsilon_n\mathbf{B}^2},$$

$$\mathbf{B} \equiv (B_2, B_3), \qquad \mathbf{u} = (U_2, U_3), \qquad \vec{\Pi} = (\Pi_2, \Pi_3). \tag{7}$$

The quantities $\mathbf{B}$ and $\vec{\Pi}$ are arbitrary functions of the Lagrangian variable ψ. In what follows everywhere (except for specially stipulated cases) the discussion is carried out in the Lagrangian variables (ψ, τ).

Passing in relations (5) to the independent variables (ψ, τ), we obtain

$$\frac{\partial x}{\partial \psi} = \frac{\Omega}{n}, \qquad \frac{\partial x}{\partial \tau} = v. \tag{8}$$

Eliminating the coordinate x from (8) and passing in the first component of the equation of motion (2) to the independent variables (ψ, τ), with the aid of (6) and (7) we arrive at the following two equations:

$$\frac{\partial v}{\partial \psi} = \frac{\partial}{\partial \tau} \frac{\sqrt{1-v^2}}{n\sqrt{1+\mathbf{u}^2}},$$
$$\frac{\partial}{\partial \tau} \frac{(\varepsilon_n + n\mathbf{B}^2)\sqrt{1+\mathbf{u}^2}\, v}{\sqrt{1-v^2}} = -\frac{\partial}{\partial \psi} n^2 \left[\left(\frac{\varepsilon}{n}\right)_n + \frac{\mathbf{B}^2}{2} + \frac{(\vec{\Pi}\mathbf{B})^2}{2\varepsilon_n^2} \right]. \tag{9}$$

Thus, in the case of two cyclic coordinates the problem has been reduced to the solution of a system of two quasilinear equations (9) for determining the quantities n and v. Having found them, with the aid of (6) and (7) we determine the remaining physical quantities. We note that the basic relations include the transverse components $\mathbf{u}$ of the velocity vector $\mathbf{U}$, depending on the density n and the Lagrangian variable ψ. In the case of one-dimensional ($\vec{\Pi} = \mathbf{u} = 0$) relativistic magnetic gas dynamics, Riemann waves were studied by K. P. Staniukovich et al. (^{4,5,9}).

Putting $\mathbf{B} = \mathbf{H} = 0$ in (7) and (9), we obtain the relations for ordinary relativistic gas dynamics

$$\frac{\partial}{\partial \tau} \frac{\varepsilon_n \sqrt{1+\mathbf{u}^2}\, v}{\sqrt{1-v^2}} = -\frac{\partial n^2 (\varepsilon/n)_n}{\partial \psi}, \qquad \frac{\partial v}{\partial \psi} = \frac{\partial}{\partial \tau} \frac{\sqrt{1-v^2}}{n\sqrt{1+\mathbf{u}^2}}, \qquad \mathbf{u} = \frac{\vec{\Pi}}{\varepsilon_n}. \tag{10}$$

In equations (10) of ordinary relativistic gas dynamics, as also in relations (9) of magnetic relativistic gas dynamics, there are transverse components $u\ (n, \psi)$ of the velocity vector $\mathbf{U}$. In the case of ordinary one-dimensional ($\vec{\Pi} = u = 0$) relativistic gas dynamics, Riemann waves were studied by K. P. Staniukovich et al. ($^{4-6,\,8-15}$).

2. Equations of gas dynamics in the variables (P, ψ, Θ). The system (9) can be represented in the form

$$\frac{\partial}{\partial \tau} W \operatorname{sh}\Theta = -\frac{\partial P}{\partial \psi}, \qquad \frac{\partial \operatorname{th}\Theta}{\partial \psi} = \frac{\partial}{\partial \tau}\left(\frac{\partial W}{\partial P}\frac{1}{\operatorname{ch}\Theta}\right), \qquad \operatorname{th}\Theta = v, \tag{11}$$

$$W(P,\psi) = (\varepsilon_n + n\mathbf{B}^2)\sqrt{1+\mathbf{u}^2}, \qquad P(n,\psi) = n^2\left[\left(\frac{\varepsilon}{n}\right)_n + \frac{\mathbf{B}^2}{2} + \frac{(\vec{\Pi}\mathbf{B})^2}{2\varepsilon_n^2}\right]. \tag{12}$$

The unknowns in (11) are the quantities P and Θ. The "effective energy" W, as a function of P and ψ, is defined parametrically by relations (12). We note that the variables (P,ψ,Θ) and the variables of L. I. Sedov (^{1}) possess many common properties.

Putting $\partial(P,\Theta)\neq 0$, let us pass in (8) and (11) to the independent variables P and Θ. With the aid of the relations obtained, we compose the total differentials dx and $d\tau$. Adding and subtracting the expressions for these differentials, we find

$$d(x\pm\tau) = e^{\pm\Theta}\left[\left(\frac{\partial W}{\partial P}\frac{\partial\psi}{\partial P} \pm \frac{\partial^2 W}{\partial P^2}\frac{\partial\psi}{\partial\Theta}\right)dP + \left(\frac{\partial W}{\partial P}\frac{\partial\psi}{\partial\Theta} \mp W\frac{\partial\psi}{\partial P}\right)d\Theta\right]. \tag{13}$$

We note that relations (13) are analogous to the relations between the differentials of the coordinates and the differentials of the variables of L. I. Sedov in the theory of steady motions (^{1,2}). The condition that on the left in (13) there stand total differentials has the form

$$W\frac{\partial^2\psi}{\partial P^2} + \frac{\partial W}{\partial\psi}\left(\frac{\partial\psi}{\partial P}\right)^2 + 2\frac{\partial W}{\partial P}\frac{\partial\psi}{\partial P} + \frac{\partial^2 W}{\partial P^2}\frac{\partial^2\psi}{\partial\Theta^2} + \frac{\partial^3 W}{\partial\psi\,\partial P^2}\left(\frac{\partial\psi}{\partial\Theta}\right)^2 = 0. \tag{14}$$

Thus, when transverse components of the velocity and magnetic field are present, there is a system of equations (11), analogous to the system of equations (11) –(12) of one-dimensional relativistic gas dynamics obtained by I. S. Shikin (^{8}). In hydrodynamics I. S. Shikin established a general analogy between unsteady relativistic and steady nonrelativistic motions and derived a number of consequences (a correspondence between the linear equation of the theory of unsteady relativistic motions and S. A. Chaplygin' s equation, etc.). Owing to this analogy, from the system (11) one obtains the nonlinear partial differential equation of second order (14), coinciding, up to notation, with L. I. Sedov' s equation in the form of Yu. V. Rudnev (^{1,2}). The coefficients of this equation depend on one of the independent variables (P) and on the unknown function (ψ).

If the "effective energy" depends only on the total pressure, $W = W(P)$, then we shall call the medium **quasibarotropic**. We shall next dwell on the analysis of such media. In the case of a quasibarotropic medium, equation (14) becomes linear

$$\frac{\partial}{\partial P}\left(W^2\frac{\partial\psi}{\partial P}\right)+WW''\frac{\partial^2\psi}{\partial\Theta^2}=0.$$

A prime everywhere denotes differentiation with respect to its argument. Solving this equation, we obtain

$$W^2\frac{\partial\psi}{\partial P}=-\frac{\partial\varphi}{\partial\Theta},\qquad WW''\frac{\partial\psi}{\partial\Theta}=\frac{\partial\varphi}{\partial P}. \tag{15}$$

The linear system (15) serves to determine the functions ψ and φ. With the aid of (15), we bring relations (13) to the form

$$d(x\pm\tau)=e^{\pm\Theta}\left(W'\,d\psi\pm\frac{1}{W}\,d\varphi\right).$$

Finding from this the differentials dx and dt and eliminating Θ with the aid of (11), we obtain

$$dx=\frac{WW'\,d\psi+v\,d\varphi}{W\sqrt{1-v^2}},\qquad dt=\frac{WW'v\,d\psi+d\varphi}{W\sqrt{1-v^2}}.$$

3. Riemann waves.

If $W''<0$, then the system (15) is of hyperbolic type; its characteristics are real. Taking in this case linear combinations of equations (11), we obtain

$$\left\{\frac{\partial}{\partial\tau}+\left[W'\,\mathrm{sh}\,\Theta+(-1)^R\sqrt{-WW''}\,\mathrm{ch}\,\Theta\right]^{-1}\frac{\partial}{\partial\psi}\right\}I_R=0,$$

$$I_R\equiv\Theta+(-1)^R\int\sqrt{-W''/W}\,dP\qquad(R=0,1).$$

Here I_R are the Riemann invariants. Passing from the Lagrange variables (ψ,τ) to the independent Euler variables (x,t) and expressing Θ through v with the aid of (11), we obtain

$$\left[\frac{\partial}{\partial t}+\frac{v+(-1)^RW'/\sqrt{-WW''}}{1+(-1)^RvW'/\sqrt{-WW''}}\,\frac{\partial}{\partial x}\right]J_R=0,$$

$$J_R\equiv\frac{v+(-1)^R\,\mathrm{th}\!\int\sqrt{-W''/W}\,dP}{1+(-1)^Rv\,\mathrm{th}\!\int\sqrt{-W''/W}\,dP}\qquad(R=0,1).$$

Here J_R are the Riemann invariants. The dependence of the Riemann invariants and of the velocities of propagation of disturbances on the transverse components and on the velocity U enters through the "effective energy."

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Note: Figure translations are in progress. See original paper for figures.