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ON A UNIFIED THEORY OF TOPOLOGICAL SPACES, PROXIMITY SPACES, AND UNIFORM SPACES

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The purpose of the present note is to outline a certain scheme for constructing a unified theory of topological spaces, proximity spaces, and uniform spaces. As is known, such a unified theory had already been constructed by Čech ([6]). Here, however, we follow another path, one closer to the ideas of Weil ([2]).

Let X be an arbitrary set, $\mathfrak{P}(X)$ the set of all its subsets, and $\mathfrak{F}(X)$ the set of all one-point subsets of the set X. If $\mathfrak{F}(X) \subset \mathfrak{M} \subset \mathfrak{P}(X)$, then by $\Phi_X(\mathfrak{M})$ we shall denote the set of all mappings of the set $\mathfrak{M}$ into $\mathfrak{P}(X)$. If $U \in \Phi_X(\mathfrak{M})$ and $V \in \Phi_X(\mathfrak{M})$, then by $U \cap V$ we denote the mapping defined by the equality $[U \cap V](A) = U(A) \cap V(A)$ for any $A \in \mathfrak{M}$.

I. We shall say that on the set X a certain **topology in the broad sense** or, more precisely, a **topology relative to the set** $\mathfrak{M}$, where $\mathfrak{F}(X) \subset \mathfrak{M} \subset \mathfrak{P}(X)$, has been introduced if a certain subset Σ of the set $\Phi_X(\mathfrak{M})$ is given that satisfies the following conditions:

1) If $U \in \Sigma$, then $A \subset U(A)$ for all $A \in \mathfrak{M}$.
2) If $U \in \Sigma$ and $V \in \Sigma$, then for any $A \in \mathfrak{M}$ there exists a certain (depending on A) $W \in \Sigma$ for which $W(A) \subset U(A) \cap V(A)$.
3) If $U \in \Sigma$, then for each $A \in \mathfrak{M}$ one can find such a (depending on A) $V \in \Sigma$ that for any $B \in \mathfrak{M}$, $B \subset V(A)$, there exists a certain (depending on B) $W \in \Sigma$ for which $W(B) \subset U(A)$.
4) If $U \in \Phi_X(\mathfrak{M})$ and if for any $A \in \mathfrak{M}$ one can find such a (depending on A) $V \in \Sigma$ that $V(A) \subset U(A)$, then $U \in \Sigma$.

In view of condition 4), it is clear that condition 2) may be replaced by the following, simpler condition:

2′) If $U \in \Sigma$ and $V \in \Sigma$, then $U \cap V \in \Sigma$.

Let Σ be a topology in X relative to some $\mathfrak{M}$, and let $\Sigma' \subset \Sigma$. We shall call Σ' a **base** of the topology Σ if, whenever $U \in \Sigma$, for any $A \in \mathfrak{M}$ one can find such a (depending on A) $V \in \Sigma'$ that $V(A) \subset U(A)$. It is easy to see that a given subset Σ of the set $\Phi_X(\mathfrak{M})$ is a base of some topology in X relative to $\mathfrak{M}$ if and only if it satisfies conditions 1), 2), and 3).

All possible topologies introduced on the given set X relative to one and the same $\mathfrak{M}$ can be compared with one another, saying that the topology Σ_1 is greater than the topology Σ_2 when $\Sigma_1 \supset \Sigma_2$. If a family of topologies in X relative to the given set $\mathfrak{M}$ is given, then by the **least upper bound** of this family we shall mean that topology Σ, relative to the same set $\mathfrak{M}$, which is the smallest of those topologies that are greater than all the topologies of the given family. The least upper bound is unique and always exists. To obtain it, consider the set Σ' of all mappings of the form $U_1 \cap U_2 \cap \dots \cap U_k$, where $U_1 \in \Sigma_1$, $U_2 \in \Sigma_2, \dots, U_k \in \Sigma_k$, and $\Sigma_1, \Sigma_2, \dots, \Sigma_k$ is some finite collection of topologies from the given family. Then Σ' will be a base of the desired least upper bound.

Let Σ be a topology in X relative to $\mathfrak{M}$, and let $A \in \mathfrak{M}$, $B \in \mathfrak{P}(X)$. We shall say that A is **close** to B relative to the topology Σ, and shall write $A\delta_\Sigma B$, if $U(A) \cap B \neq \emptyset$* for every $U \in \Sigma$. It is easy to establish that if $U \in \Phi_X(\mathfrak{M})$ and if $U(A) \cap B \neq \emptyset$ whenever $A\delta_\Sigma B$, where $A \in \mathfrak{M}$ and $B \in \mathfrak{P}(X)$, then $U \in \Sigma$.

We shall call a topology Σ in X relative to $\mathfrak{M}$ **symmetric** if the relation of closeness introduced above is symmetric when it is considered only between elements of the set $\mathfrak{M}$, i.e., if from $A\delta_\Sigma B$, where $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$, it follows that $B\delta_\Sigma A$. We note that if a family of symmetric topologies is given in some set relative to one and the same $\mathfrak{M}$, then its least upper bound may fail to be symmetric even in the case when the given family consists of only two topologies.

II. Let two sets X and X' be given, and let Σ be a topology in X relative to some $\mathfrak{M}$, while Σ' is a topology in X' relative to some $\mathfrak{M}'$. If f is a mapping of the set X into X', then we can extend f to a mapping of the set $\mathfrak{P}(X)$ into $\mathfrak{P}(X')$, defining, as usual, $f(A)$ as the set of all $f(x)$, where $x \in A$. We shall call the mapping f **continuous** if the following conditions are satisfied: a) if $A \in \mathfrak{M}$, then $f(A) \in \mathfrak{M}'$, and b) if $V \in \Sigma'$, then $f^{-1}Vf \in \Sigma$. It is easy to see that a superposition of a finite number of continuous mappings is also continuous.

Let now X be an arbitrary set, and X' a set in which a topology Σ' relative to some $\mathfrak{M}'$ has been introduced, and let a mapping f of the set X into X' be given. Then, if $\mathfrak{I}(X) \subset \mathfrak{M} \subset \mathfrak{P}(X)$ and if for every $A \in \mathfrak{M}$ we have $f(A) \in \mathfrak{M}'$, the set of all mappings of the set $\mathfrak{M}$ into $\mathfrak{P}(X)$ of the form $f^{-1}Vf$, where $V \in \Sigma'$, will be a base of a certain topology Σ relative to $\mathfrak{M}$—the smallest of all topologies in X relative to $\mathfrak{M}$ for which the mapping f is continuous. In this case we shall write

$$\Sigma = f^{-1}\Sigma' f.$$

III. Let a family $\{X_\alpha\}$ of sets X_α be given, and let in each X_α a topology Σ_α

be introduced relative to some $\mathfrak{M}_\alpha$. Consider the Cartesian product

$$X = \prod_\alpha X_\alpha$$

and denote by p_α the projection of the set X onto X_α. Let $\mathfrak{I}(X) \subset \mathfrak{M} \subset \mathfrak{P}(X)$, and suppose that the set $\mathfrak{M}$ is such that from $A \in \mathfrak{M}$ it follows that $p_\alpha(A) \in \mathfrak{M}_\alpha$ for all α. By the topology in the product $\prod_\alpha X_\alpha$ relative to $\mathfrak{M}$, **generated by the topologies** Σ_α, we shall mean the topology Σ which is the smallest of all topologies relative to $\mathfrak{M}$ for which all projections p_α are continuous. Such a topology will obviously be the least upper bound of the family $\{p_\alpha^{-1}\Sigma_\alpha p_\alpha\}$.

IV. Consider the case when a certain topology Σ is given in a set X relative to the set $\mathfrak{I}(X)$ of all one-point subsets of it. Then, denoting by Φ_X the set of all mappings of the set X into $\mathfrak{P}(X)$, we may regard $\Sigma \subset \Phi_X$ and therefore may formulate, in the following way, conditions 1)–4) from I that are satisfied by Σ:

T1) If $U \in \Sigma$, then $x \in U(x)$ for all $x \in X$.

T2) If $U \in \Sigma$ and $V \in \Sigma$, then $U \cap V \in \Sigma$.

T3) If $U \in \Sigma$, then for each $x \in X$ one can find such a $V \in \Sigma$ (depending on x) that for every $y \in V(x)$ there exists some $W \in \Sigma$ (depending on y) for which $W(y) \subset U(x)$.

T4) If $U \in \Phi_X$ and if for every $x \in X$ one can find such a $V \in \Sigma$ (depending on x) that $V(x) \subset U(x)$, then $U \in \Sigma$.

In this case we shall say that a certain **topology in the narrow sense** is given in the set X. As is easy to see, any topology in the broad sense introduced in X naturally generates a certain topology in the narrow sense in X.

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* By $\emptyset$ we denote the empty set.

If we now define the closure of any subset A of the set X as the set of all points close to it (in the sense of Sec. I), then it is easy to verify that all the Kuratowski axioms ([1]) are satisfied. Consequently, every topology in the narrow sense introduced in X turns X into a topological space. Conversely, if X is a topological space, then one can introduce in it such a topology in the narrow sense Σ which generates it in the manner indicated above. For this it suffices to denote by Σ the set of all mappings $U \in \Phi_X$ for which each point $x \in X$ is an interior point of the set $U(x)$.

It is easy to see that the notion of a continuous mapping introduced in Sec. II coincides, in the case of topological spaces, with the usual definition of this notion, and that the same is true of the notion of the topology of a Cartesian product introduced in Sec. III.

V. If Σ is a symmetric topology in some set X relative to the set $\mathfrak{P}(X)$ of all its subsets, then one can show that, for it, conditions 1)–4) of Sec. I are equivalent to the following conditions:

B1) If $U \in \Sigma$, then $A \subset U(A)$ for all $A \subset X$.

B2) If $U \in \Sigma$ and $V \in \Sigma$, then $U \cap V \in \Sigma$.

B3) If $U \in \Sigma$, then for every $A \subset X$ one can find such a (depending on A) $V \in \Sigma$ that $V[V(A)] \subset U(A)$.

B4) If U is a mapping of the set $\mathfrak{P}(X)$ into itself and if for every $A \subset X$ one can find such a (depending on A) $V \in \Sigma$ that $V(A) \subset U(A)$, then $U \in \Sigma$.

Let us add also the symmetry condition:

B5) If $U \in \Sigma$ and if $A \subset X$, $B \subset X$, $U(A) \cap B = \emptyset$, then there exists such a (depending on A and B) $V \in \Sigma$ for which $V(B) \cap A = \emptyset$.

It is easy to check that in this case, i.e. in the case of a symmetric topology relative to $\mathfrak{P}(X)$, the closeness relation introduced in Sec. I satisfies all the axioms of closeness ([3,4]). Thus, the set X turns into a proximity space. Conversely, if a proximity space X is given, then, considering the set of all mappings U of the set $\mathfrak{P}(X)$ into itself for which A and $X \setminus U(A)$ are far in the sense of the closeness of the space X for every $A \subset X$, we obtain a certain symmetric topology in X relative to $\mathfrak{P}(X)$, which generates the given closeness of the space X.

In the case of proximity spaces, the notion of a continuous mapping introduced in Sec. II coincides with the notion of proximally continuous mapping ([3,4]).

Let a family $\{X_\alpha\}$ of proximity spaces X_α be given, and let Σ_α be the symmetric topology corresponding to the space X_α relative to $\mathfrak{P}(X_\alpha)$. The topology Σ in the Cartesian product $X = \prod_\alpha X_\alpha$ relative to $\mathfrak{P}(X)$, generated by the topologies Σ_α (in the sense of Sec. III), generally speaking, will no longer be symmetric and, consequently, will not turn X into a proximity space. Therefore, by the closeness in X generated by the closenesses in X_α, we shall mean the closeness generated by the smallest symmetric topology $\widetilde{\Sigma}$ relative to $\mathfrak{P}(X)$ which is larger than Σ. One can show that if p_α is the projection of the set X onto X_α, and if $A \subset X$, $B \subset X$, then A and B are close relative to $\widetilde{\Sigma}$ if and only if the following condition is fulfilled: if $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{j=1}^m B_j$, then there exist such an i and such a j that, for all α, the sets $p_\alpha(A_i)$ and $p_\alpha(B_j)$ are close in the space X_α.

VI. Again denote by Φ_X the set of all mappings of the set X into $\mathfrak{P}(X)$. If $U \in \Phi_X$, then, defining $U(A)$ as the sum of all $U(x)$, where $x \in A$, we obtain a certain mapping of the set $\mathfrak{P}(X)$ into itself, which we shall also denote by U.

We shall now uniformize conditions T1)–T4) of Sec. IV, requiring that those mappings whose existence is in question in them do not depend on the choice of the point x or y. More precisely, consider a certain subset Σ of the set Φ_X, satisfying the following conditions:

P1) If $U \in \Sigma$, then $x \in U(x)$ for all $x \in X$.

P2) If $U \in \Sigma$ and $V \in \Sigma$, then $U \cap V \in \Sigma$.

P3) If $U \in \Sigma$, then there exists a $V \in \Sigma$ such that $V[V(x)] \subset U(x)$ for all $x \in X$.

P4) If $U \in \Phi_X$ and if there exists a $V \in \Sigma$ such that $V(x) \subset U(x)$ for all $x \in X$, then $U \in \Sigma$.

In this case we shall say that Σ is a **uniform topology** defined on the set X. A subset Σ' of a given uniform topology Σ in X will be called its **base** if for every $U \in \Sigma$ one can find a $V \in \Sigma'$ such that $V(x) \subset U(x)$ for all $x \in X$.

We shall call a given uniform topology Σ in X **symmetric** if the following symmetry condition is satisfied:

P5) For every $U \in \Sigma$ there exists a $V \in \Sigma$ such that if $y \in V(x)$, then $x \in U(y)$.

If a symmetric uniform topology Σ is given on the set X, then it turns X into a uniform space. Indeed, if $U \in \Sigma$, let $\widetilde{U}$ denote the set of those points $(x, y) \in X \times X$ for which $y \in U(x)$. Then it is easy to verify that the set of all $\widetilde{U}$, where $U \in \Sigma$, defines a certain uniform structure [5] on the set X. Conversely, if X is a uniform space, then one can introduce in it such a symmetric uniform topology that generates it. For this purpose, if $\widetilde{U}$ is an element of the uniform structure of the space X, define $U(x)$ as the set of those points $y \in X$ for which $(x, y) \in \widetilde{U}$, and thereby obtain a certain mapping $U \in \Phi_X$. Then the set Σ of all these mappings will be the desired symmetric uniform topology.

It is clear that if Σ is a uniform topology on the set X, then it is at the same time a base of a certain topology in the narrow sense in X. It is also easy to see that if all mappings $U \in \Sigma$ are replaced by their extensions to the whole set $\mathfrak{P}(X)$, then we obtain a base of a certain topology $\widetilde{\Sigma}$ in X relative to $\mathfrak{P}(X)$. Moreover, one can show that if Σ is symmetric and, consequently, generates a certain uniform space, then $\widetilde{\Sigma}$ will also be symmetric, so that it will generate a certain proximity space.

Finally, let us note that the notion of a uniformly continuous mapping, as well as the notion of the uniform topology of a Cartesian product, are introduced in a natural way in the spirit of the preceding exposition and coincide in the symmetric case with the corresponding notions in the theory of uniform spaces. If, however, wishing to introduce the notion of a uniform topology in the broad sense, we uniformize conditions 1)–4) of Section I, then we obtain a certain system of conditions equivalent, with respect to the notion of a base, to the system of conditions P1)–P4), so that, in the final analysis, we obtain essentially one single notion of a uniform topology.

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Note: Figure translations are in progress. See original paper for figures.