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Abstract

Full Text

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ON THE APPROXIMATION OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR PARABOLIC AND HYPERBOLIC EQUATIONS BY MEANS OF SOLUTIONS OF THE CAUCHY PROBLEM

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In the present paper, solutions of boundary-value problems for parabolic and hyperbolic equations of second order are obtained as limits, as $\varepsilon \to 0$, of solutions of the Cauchy problem for certain equations with coefficients depending on ε. Similar problems were considered in works $(^{1-3})$, etc.

1. In the cylinder $Q^- = \{\Omega^- \times [0T]\}$, where Ω^- is a bounded domain of the space E_n with boundary Γ, a parabolic equation is given

$$a(x,t)\frac{\partial u}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} c_i(x,t)\frac{\partial u}{\partial x_i} + d(x,t)u = \varphi(x,t). \quad (1)$$

Let $a, b_{ij}, c_i, d, \varphi, \partial a/\partial t, \partial b_{ij}/\partial t, \partial d/\partial t$ $(i,j = 1,2,\dots,n)$ be bounded measurable functions, $d(x,t) \leqslant 0$,

$$a(x,t) \leqslant \bar{a} < 0, \qquad \bar{\lambda}_1 \sum_{i=1}^{n} \xi_i^2 \geqslant \sum_{i,j=1}^{n} b_{ij}(x,t)\xi_i\xi_j \geqslant \lambda_1 \sum_{i=1}^{n} \xi_i^2 \quad (2)$$

for any real vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $\bar{\lambda}_1 \geqslant \lambda_1 > 0$. The boundary of the cylinder $S = \{\Gamma \times [0T]\}$ belongs to the class A_2^2 (see $(^4)$). We introduce, as usual, the functional spaces $W_2^{1,1}(Q^-)$, $\overset{0}{W}{}_2^{1,1}(Q^-)$, $W_2^1(\Omega^-)$, $\overset{0}{W}{}_2^1(\Omega^-)$ (see $(^{5,6})$, p. 93).

For equation (1) the following problems in Q^- are considered:

1) the first boundary-value problem with conditions

$$u(x,t)\big|_{t=0} = f(x), \qquad f(x) \in \overset{0}{W}{}_2^1(\Omega^-); \quad (3)$$

$$u(x,t)\big|_S = 0; \tag{4}$$

2) the second boundary-value problem with condition (3) and

$$\left.\frac{\partial u}{\partial \nu}\right|_S = 0, \qquad \text{where } \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} b_{ij}\frac{\partial u}{\partial x_j}\cos(nx_i); \tag{5}$$

3) the third boundary-value problem with condition (3) and

$$\left.\frac{\partial u}{\partial \nu} + Fu\right|_S = 0, \tag{6}$$

where n is the outward normal to S; $F(x,t) > 0$, $F \subset C^1$ in $\{E_n \times [0T]\}$.

In the case of problems (1), (3), (5) it is also assumed that the functions $\partial a/\partial x_k$, $\partial b_{ij}/\partial x_k$, $\partial c_i/\partial x_k$, $\partial d/\partial x_k$, $\partial \varphi/\partial x_k$ $(i,j,k = 1,2,\dots,n)$ are measurable and bounded.

By a generalized solution of problem (1), (3), (4) we shall mean a function $u(x,t) \in \overset{0}{W}{}_2^{1,1}(Q^-)$ such that $u(x,0) = f(x)$ and, for any function $\Phi(x,t) \in \overset{0}{W}{}_2^{1,1}(Q^-)$, $\Phi(x,T) = 0$, the integral identity

$$\Psi(u,\Phi) \equiv \iint_{Q^-}\left[a\frac{\partial u}{\partial t}\Phi - \sum_{i,j=1}^{n} b_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i} + \sum_{i=1}^{n} c_i\frac{\partial u}{\partial x_i}\Phi + du\Phi - \varphi\Phi\right] dx\,dt = 0. \tag{7}$$

By a generalized solution of problem (1), (3), (5) we shall mean a function $u(x,t) \in W_2^{1,1}(Q^-)$ such that $u(x,0) = f(x)$ and, for any function $\Phi(x,t) \in W_2^{1,1}(Q^-)$, $\Phi(x,T) = 0$, the integral identity (7) is satisfied.

By a generalized solution of problem (1), (3), (6) we shall mean a function $u(x,t) \in W_2^{1,1}(Q^-)$ such that $u(x,0) = f(x)$ and, for any function $\Phi(x,t) \in W_2^{1,1}(Q^-)$, $\Phi(x,T) = 0$, the integral identity is satisfied:

$$\Psi(u,\Phi) - \int_S Fu\Phi\,ds = 0. \tag{8}$$

Let us extend the coefficients of equation (1) to $Q^+ = \{\Omega^+ \times [0T]\}$, where $\Omega^+ = E_n/\overline{\Omega^-}$, as functions (t,x,ε) in such a way that, in the case of problem (1), (3), (5),

$$|a| \le K\varepsilon^{\alpha}, \quad |\partial a/\partial t| \le K\varepsilon^{\alpha}, \quad |b_{ij}| \le K\varepsilon^{\beta}, \quad |\partial b_{ij}/\partial t| \le K\varepsilon^{\beta}, \tag{9}$$

$$|c_i| \le K\varepsilon^{\gamma}, \quad |d| \le K\varepsilon^{\delta}, \quad |\partial d/\partial t| \le K\varepsilon^{\delta}, \quad |\varphi| \le K\varepsilon^{\eta} \quad (i,j = 1,2,\dots,n),$$

where $\varphi(x,t,\varepsilon)=0$ outside a certain finite domain, $d(x,t,\varepsilon)\le 0$,

$$\eta\ge\alpha/2,\quad \gamma\ge(\beta+\alpha)/2,\quad \alpha\le\beta, \tag{10}$$

the constant K does not depend on ε,

$$a(x,t,\varepsilon)\le\tilde a(\varepsilon)<0,\quad \varepsilon^{\beta}\bar\lambda_2\sum_{i=1}^{n}\xi_i^2\ge\sum_{i,j=1}^{n}b_{ij}(x,t,\varepsilon)\xi_i\xi_j\ge\varepsilon^{\beta}\lambda_2\sum_{i=1}^{n}\xi_i^2,$$

$$\bar\lambda_2\ge\lambda_2>0. \tag{11}$$

For problem (1), (3), (5), we assume, in addition, that in Q^+

$$|\partial a/\partial x_k|\le K\varepsilon^{\alpha},\quad |\partial b_{ij}/\partial x_k|\le K\varepsilon^{\beta},\quad |\partial c_i/\partial x_k|\le K\varepsilon^{\gamma},$$

$$|\partial d/\partial x_k|\le K\varepsilon^{\delta},\quad |\partial\varphi/\partial x_k|\le K\varepsilon^{\eta}. \tag{12}$$

For problem (1), (3), (6) the coefficients of equation (1) are extended to Q^+ so that inequalities (9), (10), (11) are satisfied. In doing so we put

$$d(x,t,\varepsilon)=-F^2(x,t)\varepsilon^{-\beta};\qquad b_{ij}(x,t,\varepsilon)=\varepsilon^{\beta}\quad\text{for } i=j;$$

$$b_{ij}(x,t,\varepsilon)=0\quad\text{for } i\ne j. \tag{13}$$

By $A^h, B_{ij}^h, C_i^h, D^h, \Phi^h$ we denote the averages of the thus obtained discontinuous functions $a, b_{ij}, c_i, d, \varphi$ with averaging radius $h=\varepsilon^{|\delta|}$ for problem (1), (3), (4), $h=\varepsilon^{|\beta|}$ for problem (1), (3), (5), and $h=\varepsilon^{\mu}$, $\mu>|\beta|$, for problem (1), (3), (6).

In the case of problem (1), (3), (5) the averaging is carried out in a special manner. We cover $\overline{Q^-}$ by a finite number of domains R_i $(i=1,2,\dots,N)$ such that in R_k containing points $S=\{\Gamma\times[0T]\}$ it is possible to pass to coordinates $t, y_1^k,\dots,y_n^k$, $y_i^k=\varphi_i(x_1,\dots,x_n)$ $(i=1,2,\dots,n)$, in which a piece of the boundary S is transformed into a piece of the plane $y_n^k=0$, and in $R_\ell\cap R_k$

$$y_n^k=y_n^\ell.$$

Denote

$$R_{N+1}=\{E_n\times[0T]\}-\sum_{i=1}^{N}R_i.$$

Let $\chi_k(x)$ be smooth functions, $\chi_k(x)=0$ outside R_k, $0\le\chi_k\le 1$,

$$\sum_{k=1}^{N+1}\chi_k=1.$$

Put:

$$A^h=\sum_{k=1}^{N+1}\chi_k(a)_k^h,\qquad B_{ij}^h=\sum_{k=1}^{N+1}\chi_k(b_{ij})_k^h,\qquad C_i^h=\sum_{k=1}^{N+1}\chi_k(c_i)_k^h,$$

$$D^h = \sum_{k=1}^{N+1} \chi_k (d)_k^h, \qquad \Phi^h = \sum_{k=1}^{N+1} \chi_k (\varphi)_k^h,$$

where $(a)_k^h$, $(b_{ij})_k^h$, $(c_i)_k^h$, $(d)_k^h$, $(\varphi)_k^h$ denote the averages of the corresponding functions in the domain R_k with respect to the variables $t, y_1^k, \dots, y_n^k$.

Consider the Cauchy problem for the equation

$$A^h(x,t,\varepsilon)\frac{\partial u}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(B_{ij}^h(x,t,\varepsilon)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} C_i^h(x,t,\varepsilon)\frac{\partial u}{\partial x_i} +$$

$$+D^h(x,t,\varepsilon)u = \Phi^h(x,t,\varepsilon) \tag{14}$$

with the condition

$$u(x,t)\big|_{t=0} = f^h(x), \tag{15}$$

where $f^h(x) \in \overset{0}{C}{}^{(\infty)}(\Omega^-)$, $f^h(x) = 0$ outside Ω^-, and

$$\|f^h(x) - f(x)\|_{\overset{0}{W}{}_2^1(\Omega^-)} \to 0$$

as $h \to 0$.

For the solution of problem (14), (15), estimates have been obtained, uniform in h and ε, by means of which the following is proved.

Theorem 1. The solution u_ε^h of problem (14), (15) converges as $\varepsilon \to 0$ in the mean in the domain Q_σ^-, for any $\sigma > 0$, to the generalized solution u_0 of problem (1), (3), (4), if $\delta \leqslant \alpha$, $\beta < -\delta$, $\delta < 0$; to the generalized solution of problem (1), (3), (5), if $\delta \leqslant \alpha$, $\beta > -\delta$, $\beta > 0$; to the generalized solution of problem (1), (3), (6), if $-\beta < \alpha$, $\beta = -\delta$, $\beta > 0$;

2. Suppose that in the cylinder Q^- there is given the hyperbolic equation

$$a(x,t)\frac{\partial^2 u}{\partial t^2} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} c_i(x,t)\frac{\partial u}{\partial x_i} +$$

$$+g(x,t)\frac{\partial u}{\partial t} + d(x,t)u = \varphi(x,t). \tag{16}$$

The functions $a, b_{ij}, c_i, g, d, \varphi, \partial a/\partial t, \partial b_{ij}/\partial t, \partial d/\partial t$ $(i,j = 1,2,\dots,n)$ are bounded and measurable; $a(x,t)$ is a continuous function. The inequalities (2) are satisfied, $S \subset A^2$. For equation (16) consider mixed problems with the initial conditions:

$$u(x,t)\big|_{t=0} = f_0(x), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = f_1(x), \tag{17}$$

where $f_0(x) \in \overset{0}{W}{}_2^1(\Omega^-)$, $f_1(x) \in L_2(\Omega^-)$, and 1) the boundary condition of the first kind (4); 2) the boundary condition of the second kind (5); 3) the boundary condition of the third kind (6).

For problem (16), (17), (5), the functions $\partial a/\partial x_k$, $\partial b_{ij}/\partial x_k$, $\partial c_i/\partial x_k$, $\partial g/\partial x_k$, $\partial d/\partial x_k$, $\partial \varphi/\partial x_k$, $\partial c_i/\partial t$, $\partial g/\partial t$, $\partial \varphi/\partial t$, $\partial^2 b_{ij}/\partial t^2$ $(i,j,k = 1,2,\dots,n)$ are also bounded and measurable, $S \subset A^3$, $f_0(x) \in \overset{0}{W}{}_2^2(\Omega^-)$, $f_1(x) \in \overset{0}{W}{}_2^1(\Omega^-)$, where $\overset{0}{W}{}_2^2(\Omega^-)$ is the space obtained by completing the functions of the class $C_0^{(\infty)}(\Omega^-)$ in the norm W_2^2.

By a generalized solution of problem (16), (17), (4) we shall mean a function $u(x,t) \in \overset{0}{W}{}_2^{1,1}(Q^-)$ such that $u(x,0) = f_0(x)$ and, for any function $\Phi(x,t) \in \overset{0}{W}{}_2^{1,1}(Q^-)$, $\Phi(x,T) = 0$, the integral identity holds:

$$\widetilde{\Psi}(u,\Phi) \equiv \iint_{Q^-} \left[-a\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t} - \sum_{i,j=1}^{n} b_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i} + \sum_{i=1}^{n} c_i \frac{\partial u}{\partial x_i}\Phi - \frac{\partial a}{\partial t}\frac{\partial u}{\partial t}\Phi + \right.$$

$$\left. + g\frac{\partial u}{\partial t}\Phi + du\Phi - \varphi\Phi \right] dx\,dt - \int_{\Omega^-} a(x,0)\Phi(x,0)f_1(x)\,dx = 0. \tag{18}$$

By a generalized solution of problem (16), (17), (5) we shall mean a function $u(x,t) \in W_2^{1,1}(Q^-)$ such that $u(x,0) = f_0(x)$ and, for any function $\Phi(x,t) \in W_2^{1,1}(Q^-)$, $\Phi(x,T) = 0$, the integral identity (18) is satisfied.

By a generalized solution of problem (16), (17), (6) we shall mean a function $u(x,t) \in W_2^{1,1}(Q^-)$ such that $u(x,0) = f_0(x)$ and, for any function $\Phi(x,t) \in$ $\in W_2^{1,1}(Q^-)$, $\Phi(x,T) = 0$, the integral identity

$$\widetilde{\Psi}(u,\Phi) - \int_S Fu\Phi\,ds = 0. \tag{19}$$

We extend the functions $a, b_{ij}, c_i, g, d, \varphi$ to Q^+ for problem (16), (17), (4) so that inequalities (9), (11) and

$$|g(x,t,\varepsilon)| \leqslant K\varepsilon^{\omega}, \tag{20}$$

are satisfied; for problem (16), (17), (6), conditions (9), (11), (13), (20); for problem (16), (17), (5), inequalities (9), (11), (12), (20) and

$$|\partial^2 b_{ij}/\partial t^2| \leqslant K\varepsilon^{\beta}, \quad |\partial c_i/\partial t| \leqslant K\varepsilon^{\gamma}, \quad |\partial g/\partial t| \leqslant K\varepsilon^{\omega},$$
$$|\partial \varphi/\partial t| \leqslant K\varepsilon^{\eta}, \quad |\partial g/\partial x_i| \leqslant K\varepsilon^{\omega} \quad (i, j = 1, 2, \dots, n).$$

$A^h, B^h_{ij}, C^h_i, G^h, D^h, \Phi^h$ denote the averages of the corresponding discontinuous functions $a, b_{ij}, c_i, g, d, \varphi$ with averaging radius: $h = \varepsilon^{|\delta|}$ for problem (16), (17), (4); $h = \varepsilon^{|\beta|}$ for problem (16), (17), (5); $h = \varepsilon^{\mu}$, $\mu > |\beta|$, for problem (16), (17), (6). In the case of problem (16), (17), (5), the averaging is carried out as for problem (1), (3), (5).

The exponents of powers of ε satisfy the conditions

$$\gamma \geqslant (\beta + \alpha)/2, \quad \eta \geqslant \alpha/2, \quad \beta > \alpha, \quad \omega \geqslant \alpha.$$

Consider the Cauchy problem for the equation

$$A^h(x,t,\varepsilon)\frac{\partial^2 u}{\partial t^2} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(B^h_{ij}(x,t,\varepsilon)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} C^h_i(x,t,\varepsilon)\frac{\partial u}{\partial x_i}$$
$$+G^h(x,t,\varepsilon)\frac{\partial u}{\partial t} + D^h(x,t,\varepsilon)u = \Phi^h(x,t,\varepsilon) \tag{21}$$

with initial conditions

$$u(x,t)\Big|_{t=0} = f^h_0(x), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = f^h_1(x), \tag{22}$$

where

$$f^h_0(x), \quad f^h_1(x) \in \overset{0}{C}{}^{(\infty)}(\Omega^-), \qquad f^h_0(x) = f^h_1(x) = 0 \quad \text{outside } \Omega^-;$$
$$\|f^h_0(x) - f_0(x)\|_{\overset{0}{W}{}^1_2(\Omega^-)} + \|f^h_1(x) - f_1(x)\|_{L_2(\Omega^-)} \to 0 \quad \text{as } h \to 0.$$

Theorem 2. The solution u_ε of problem (21), (22) converges as $\varepsilon \to 0$, in the mean in the domain Q_σ for $\sigma > 0$, to the generalized solution u_0 of problem (16), (17), (4), if $\alpha \leqslant \delta$, $\beta < -\delta$, $\delta < 0$; to the generalized solution of problem (16), (17), (6), if $\alpha > -\beta$, $\beta = -\delta$, $\beta > 0$; to the generalized solution of problem (16), (17), (5), if $\alpha \leqslant \delta$, $\beta > -\alpha$, $\beta > 0$ and

$$\|f^h_0(x) - f_0(x)\|_{\overset{0}{W}{}^2_2(\Omega^-)} + \|f^h_1(x) - f_1(x)\|_{\overset{0}{W}{}^1_2(\Omega^-)} \to 0 \quad \text{as } h \to 0.$$

If $\beta > 0$, $\alpha > 0$, $\gamma > 0$, $\delta > 0$, $\eta > 0$, $\omega > 0$, then for problem (16), (17), (5) Theorem 2 is true under the condition that $f_0(x) \in \overset{0}{W}{}_2^1(\Omega^-)$, $f_1(x) \in L_2(\Omega^-)$.

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Note: Figure translations are in progress. See original paper for figures.