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Abstract

Full Text

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SINGULAR INTEGRAL EQUATIONS IN LIPSCHITZ SPACES

(Presented by Academician V. I. Smirnov on 28 V 1964)

In the paper ([1]) S. G. Mikhlin (see also ([2])) investigated multidimensional singular integral equations in Lipschitz spaces. The main result of that paper consists in the fact that, under certain smoothness conditions imposed on the data of the equation, every solution of it satisfies a Lipschitz condition of order $\alpha \in (0,1)$. However, this result was obtained under rather restrictive assumptions. For example, if the characteristic does not depend on the pole, it is required that it have square-summable generalized derivatives of order l on the unit sphere, where $l \geq m/2+1$ for even m and $l \geq (m+1)/2+2$ for odd m (m is the dimension of the space). In the case of a characteristic depending on the pole, additional restrictions arise.

The above-mentioned result of S. G. Mikhlin was obtained on the basis of his theory of singular equations in L_p. The purpose of the present note is to construct a general theory of singular equations on manifolds in Lipschitz spaces. As a final result, a theorem is obtained on the membership of solutions in a Lipschitz space under assumptions weaker than in ([1]).

Let Γ be an m-dimensional closed manifold of Lyapunov type, and let φ be a function on Γ whose support is mapped by means of a one-to-one smooth transformation τ onto a domain $G \subset E_m$. We shall denote by $\varphi_\tau, \varphi^\tau$ the operators defined by the equalities

$$\varphi_\tau u(x) = \varphi(\tau^{-1}(x))\, u(\tau^{-1}(x)), \qquad \varphi^\tau v(\xi) = \varphi(\xi)\, v(\tau(\xi)),$$

where ξ and x are arbitrary points on Γ and G, respectively, and $u(\xi)$ and $v(x)$ are functions given respectively on Γ and G.

By $\mathrm{Lip}_\Gamma\, \alpha$ we shall denote the space of functions given on Γ and satisfying a Lipschitz condition of order α, $0 < \alpha < 1$.

Definition 1*. An operator A is called **singular** on Γ if the following conditions are fulfilled:

1. For any functions φ and ψ from $\mathrm{Lip}_\Gamma\, \alpha$ with nonintersecting supports on Γ, the operator $\varphi A \psi$ is completely continuous in $\mathrm{Lip}_\Gamma\, \alpha$.

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2. For any functions φ and ψ from $\mathrm{Lip}_\Gamma\,\alpha$, whose supports can be mapped, by means of a single one-to-one smooth transformation τ, onto a domain of the space E_m (i.e., are situated within one coordinate system), the equality

$$\varphi A\psi = \varphi^\tau \mathcal{A}\psi_\tau + T,$$

holds, where T is an operator completely continuous in $\mathrm{Lip}_\Gamma\,\alpha$, and $\mathcal{A}$ is a singular operator in E_m, i.e.

$$\mathcal{A}u(x) = a(x)u(x) + \lim_{\varepsilon\to 0}\int_{r>\varepsilon}\frac{f(x,\theta)}{r^m}u(y)\,dy,$$

where

$$a(x) \in \mathrm{Lip}_{E_m}\,\alpha, \qquad \theta = \frac{y-x}{|y-x|}, \qquad r = |y-x|.$$

* This definition is analogous to the definition of a singular operator in $L_p(\Gamma)$ given by Seeley ([3]).

The symbol Φ_A of the operator A, by definition, coincides with the symbol of the operator $\mathcal{A}$.

In what follows we follow the notation adopted in the monograph of S. G. Mikhlin ([2]).

We formulate a theorem on the boundedness of a singular operator in $\mathrm{Lip}_\Gamma\alpha$. Such a theorem for multidimensional singular integrals was first proved by J. Giro ([4]) (see also ([2]), § 6) under considerably more restrictive conditions.

Theorem 1. If $f(x,\theta) \in \widehat{W}_1^{(1)}(S)$ and if

$$\|f(x+h,\theta) - f(x,\theta)\|_{L(S)} \le B|h|^\alpha, \tag{1}$$

then the operator A is bounded in $\mathrm{Lip}_\Gamma\alpha$.

For operators defined on a two-dimensional manifold, one can obtain a more precise result.

Theorem 1′. Let $m=2$. If (1) holds and $f(x,\theta) \in \widehat{L}(S)$, and if

$$\|f(x,\theta_\omega) - f(x,\theta)\|_{L(S)} \le B\omega^\beta, \tag{2}$$

where $\beta > \alpha$ and θ_ω is the rotation of the vector θ through a constant angle ω, then the operator A is bounded in $\mathrm{Lip}_\Gamma\alpha$.

Using a result of S. G. Mikhlin ([5]) (see also ([2]), § 31), from Theorem 1 one can obtain a criterion for the boundedness of the operator A in terms of the symbol $\Phi_A(x,\theta)$.

Theorem 2. If $\Phi_A(x,\theta) \in \widehat{W}_2^{(l)}(S)$, where $l \geq (m+2)/2$, and if

$$\|\Phi_A(x+h,\theta) - \Phi_A(x,\theta)\|_{W_2^{(l-1)}(S)} \leq B|h|^\alpha,$$

then the operator A is bounded in $\mathrm{Lip}_\Gamma \alpha$.

The following theorem on multiplication of symbols holds.

Theorem 3. Let A_1 and A_2 be singular operators on Γ. If $\Phi_{A_i}(x,\theta) \in \widehat{W}_2^{(l)}(S)$, where $l \geq (m+3)/2$, and if

$$\|\Phi_{A_i}(x+h,\theta) - \Phi_{A_i}(x,\theta)\|_{W_2^{(l-1)}(S)} \leq B|h|^\alpha \quad (i=1,2),$$

then the operator $A_1A_2 - A_2A_1$ is completely continuous in $\mathrm{Lip}_\Gamma \alpha$, and the symbol of the product A_1A_2 is equal to the product of the symbols $\Phi_{A_1}\Phi_{A_2}$.

In the proof of Theorem 3 the following two lemmas are used.

Lemma 1. Let $\Phi_A(x,\theta) \in \widehat{W}_2^{(l)}(S)$, where $l \geq (m+3)/2$, and

$$\|\Phi_A(x+h,\theta) - \Phi_A(x,\theta)\|_{W_2^{(l-1)}(S)} \leq B|h|^\alpha.$$

Then the expansion of the symbol in a series in spherical functions,

$$\Phi_A(x,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} b_n^{(k)}(x) Y_{n,m}^{(k)}(\theta),$$

corresponds to an expansion of the operator A into a series convergent in the $\mathrm{Lip}_\Gamma \alpha$ norm,

$$A = A_1^{(0)} + \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} A_n^{(k)},$$

where $A_n^{(k)}$ are singular operators on Γ corresponding to the operators $\mathcal{A}_n^{(k)}$ in E_m, where

$$\mathcal{A}_1^{(0)} u(x) = b_1^{(0)}(x) u(x), \qquad \mathcal{A}_n^{(k)} u(x) = \frac{b_n^{(k)}(x)}{\gamma_{n,m}} \int_{E_m} \frac{Y_{n,m}^{(k)}(\theta)}{r^m}\, u(y)\, dy.$$

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Lemma 2. Let φ and ψ be functions from the space $\operatorname{Lip}\Gamma\alpha$ with compact supports on Γ, and let the function $b(x)$ satisfy a Lipschitz condition of order α in E_m. Then the operator $\varphi\mathfrak{B}\psi$, where $\mathfrak{B}$ is the integral operator with kernel

$$\frac{b(y)-b(x)}{|y-x|^m}\,Y^k_{n,m}(\theta),$$

is completely continuous in $\operatorname{Lip}\Gamma\alpha$.

In the following theorem conditions are given under which every bounded solution of the equation $Au = g$, where $g \in \operatorname{Lip}\Gamma\alpha$, belongs to the space $\operatorname{Lip}\Gamma\alpha'$, $\alpha' > 0$.

Theorem 4. Let $\inf|\Phi_A(x,\theta)| > 0$, $f(x,\theta) \in \widehat{W}^{(2)}_2(S) \cap C^{(1)}(S)$; let

$$|f(x+h,\theta) - f(x,\theta)| \leqslant B|h|^\gamma \quad (\gamma > 0), \tag{3}$$

$$\|f(x+h,\theta) - f(x,\theta)\|_{W^{(1)}_2(S)} \leqslant B|h|^\alpha.$$

Then every bounded solution of the equation $Au = g$ belongs to the space $\operatorname{Lip}\Gamma\alpha'$.

For $m = 2$, using Theorem 1, this result can be strengthened.

Theorem 4′**.** Let $\inf|\Phi_A(x,\theta)| > 0$, $f(x,\theta) \in \widehat{C}^{(1)}(S)$. Suppose that (3) holds and

$$\|f(x+h,\theta) - f(x,\theta)\|_{W^{(\varepsilon)}_2(S)} \leqslant B|h|^\alpha, \tag{4}$$

where $\varepsilon > 0$. Then every bounded solution of the equation $Au = g$ belongs to the space $\operatorname{Lip}\Gamma\alpha'$.

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REFERENCES

[1] S. G. Mikhlin, DAN, **138**, No. 3 (1961).
[2] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Moscow, 1962.
[3] R. T. Seeley, Am. J. Math., **81**, No. 3 (1959).
[4] G. Giraud, Ann. Sci. École Norm. Super., **51**, f. 3, 4 (1934).
[5] S. G. Mikhlin, DAN, **126**, No. 2 (1959).

Note: Figure translations are in progress. See original paper for figures.

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