G. N. PERLATOV

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Abstract

Full Text

G. N. PERLATOV

SOME PROPERTIES OF INTERNAL EXTENSIONS OF AN INCOMPLETE ARCHIMEDEAN SPACE

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In the present note the following axioms from the system of axioms of geometry proposed by D. Hilbert are assumed to be satisfied: I_{1-6}, II_{1-4}, III_{1-5}, IV and V_1 ((*1*), pp. 56–91).

In view of the incompleteness of the spaces, it is possible to supplement the space with new elements, which we shall call "ideal," in contrast to the old "real" elements. In what follows only internal extensions of an incomplete space are considered, i.e., extensions not connected with an increase in the dimension of the space. The presence of the axioms restricts the possible mutual relations between ideal and real elements when the space is extended. The consideration of some of these relations, begun earlier by the author ((*2*)), will be continued in the present note.

§ 1. **Circles and spheres in an incomplete Archimedean space.** Let us consider some properties of ideal and real circles in an incomplete Archimedean plane.

For real circles the following theorems are proved:

Theorem 1. *No real circle can have an ideal center.*

Theorem 2. *If there exists at least one ideal point in the space, then every real circle contains an everywhere dense set of ideal points.*

With respect to ideal circles the following is proved:

Theorem 3 (existence). *If there exists at least one ideal point in the space, then in any real plane there exist three types of ideal circles: a) ideal circles without real points, but with a real center; b) ideal circles with an ideal center and a single real point; c) ideal circles with an ideal center and two real points.*

To clarify the question of the existence of ideal circles with an ideal center containing not a single real point, definitions of extensions of class I and class II are introduced.

Definition. We shall call an extension of a real plane an **extension of class I** if, under such an extension, every circle with an ideal center contains at least one real point; otherwise we shall call the extension an **extension of class II**.

Theorem 4. *In order that an extension Σ of the original coordinate field Ω be an extension of class I, it is necessary that every ideal element $\omega \in \Sigma$ be quadratic over Ω.*

Analogously one may consider not only extensions of the real plane, but also extensions of space; only instead of circles one should consider spheres, and to the accepted axioms add axioms I_{7-8}.

The following theorems, analogous to the corresponding theorems for plane extensions, are proved:

Theorem 5. *A real sphere cannot have an ideal center.*

Theorem 6. *If there exists at least one ideal point in the space, then on any real circle of a great circle of a real sphere there is contained an everywhere dense set of ideal points.*

Theorem 7. If there exists at least one ideal point in the space, then there exist three types of ideal spheres: a) ideal spheres without real points; b) ideal spheres with a single real point; c) ideal spheres with one real circumference of a small circle.

Definitions are introduced of spatial extensions of classes I and II according as, under such extensions, there do not exist or do exist ideal spheres with an ideal center that contain no real points. A necessary criterion for a spatial extension of class I is formulated and proved analogously to the necessary criterion for a planar extension.

2. Pencils of lines and bundles of planes in an incomplete Archimedean space. The study of circles and spheres with an ideal center is connected with the consideration of pencils with an ideal center.

Definition. An extension of the real plane will be called an **extension of type I** if through every ideal point of the plane there passes at least one (and therefore only one) real line. If there exist ideal points of the plane through which no real line passes, then such an extension will be called an **extension of type II**.

The following simple criterion is established for extensions of type I.

Theorem 8. In order that an extension be an extension of type I, it is necessary and sufficient that it be quadratic over the initial coordinate field.

Corollary. In order that the coordinate field Ω admit no extensions of type I, it is sufficient that Ω be real-closed.

Indeed, a real-closed field is closed with respect to extracting square roots from positive elements ([3]).

The question of the existence of coordinate fields not closed with respect to extracting square roots from positive elements was answered positively by D.

Hilbert, who showed that not every problem solvable by compass and straightedge can also be solved by straightedge and a standard unit of length (([1]), pp. 185-186).

The connection between extensions of class I and extensions of type I is established in the following two theorems:

Theorem 9. Every extension of class I is an extension of type I.

Theorem 10. No extension of type I is an extension of class I.

As a consequence of these theorems, the class of extensions of class I is empty. In other words: under every extension of the plane there exist ideal circles with an ideal center and without real points. The specific properties of extensions of type II are formulated in the theorems:

Theorem 11. The set of points of an incomplete Archimedean plane admitting an extension of type II can be ordered and metrized in such a way that, in the new metrization, the ordering is Archimedean (such an ordering is based on the fact that any sphere with center at an ideal point of type II contains no more than one real point).

Theorem 12. In the case of a real plane admitting an extension of type II, there can be constructed a group of transformations G of the extended plane satisfying the conditions: a) any nonidentity transformation from G carries an arbitrary configuration of real points and lines into a configuration of ideal points and lines; b) there exist ideal lines which are not carried by any transformation from G into real lines.

Definition. An ideal point of an extension of the plane will be called an **ideal point of type I** or an **ideal point of type II** according as at least one real line passes through this point or not.

Through an arbitrary ideal point of type I there passes one and only one real line, i.e., a field is established of real directions passing through points of type I.

Consider an arbitrary real line MN, which we shall call the axis, and an arbitrary ideal point of type I, K, outside MN. Let $KL \perp MN$, and let KT be the unique real line passing through K, and $\angle LKT = \varphi$. We shall call the ideal point K a φ-point, or a point of the φ-class with respect to MN.

Theorem 13. *The ideal points of any real line, except for the axis MN, belong to one class of φ-points. The points of the axis MN do not belong to any φ-class. On any ideal line not parallel to any real line, there is an everywhere dense set of ideal points of any one and the same φ-class. On any ideal line parallel to some real line l, there is an everywhere dense set of ideal points of any φ-class, except for the points of the φ-class to which the points of the line l belong.*

With respect to ideal points of type II, an analogous result is proved:

Theorem 14. *If in some extension of the real plane there is at least one ideal point of type II, then in this extension, on every ideal line there is an everywhere dense set of ideal points of type II.*

For the proof of the theorem, it is first shown that in a rectangle with sides parallel to the coordinate axes and with vertices at ideal points, if one of the vertices is an ideal point of type II, then at least one more of the vertices is also an ideal point of type II. Next it is proved that on any ideal line parallel to one of the coordinate axes, there is an everywhere dense set of ideal points of type II. After this, the theorem is extended to oblique ideal lines with real and ideal slopes.

Extensions of space are considered analogously. An ideal point of an extension of space is called a point of type I or II according as at least one real plane passes through this point or not. An extension of space is called an extension of type I or type II according as such an extension does not contain, or contains, ideal points of type II. An ideal point of type I is called a point of subtype I or II according as exactly one real plane, or more than one real plane, passes through this point.

We indicate the following theorems for spatial extensions:

Theorem 15. *In order that an extension of space be an extension of type I, it is necessary and sufficient that it be quadratic over the original coordinate field.*

Theorem 16. *In an extension of space of type I, all ideal points are ideal points of type I of subtype I.*

Theorem 17. *In the case of a spatial extension of type II, there exist ideal spheres containing two and only two real points.*

It should be noted that the proofs of the theorems indicated in the present note are carried out within the limits of Hilbert' s "finite framework," i.e., without the use of the apparatus of set theory. The expression "a set of points with a given property everywhere dense on some line," occurring in the formulations of a number of theorems, may be replaced by an expression in the spirit of the "finite framework" : "every segment of the line contains at least one point with the given property."

Kaluga State
Pedagogical Institute

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3. B. L. van der Waerden, *Modern Algebra*, Part 1, 1934, pp. 222–223.

Note: Figure translations are in progress. See original paper for figures.