P. E. SOBOLEVSKII

1964

SovietRxiv

Abstract

Full Text

P. E. SOBOLEVSKII

COERCIVITY INEQUALITIES FOR ABSTRACT PARABOLIC EQUATIONS

(Presented by Academician I. N. Vekua, 4 II 1964)

1. The problem considered is

$$\frac{dv}{dt} + Av = f(t), \qquad v(0) = v_0 \tag{1}$$

in a Banach space E. Here $v(t)$ and $f(t)$ are the unknown and the given functions, defined on $[0, T]$, with values in E; dv/dt is the derivative, understood as the limit in the norm of E of the corresponding difference quotient; $(-A)$ is the infinitesimal operator of the strongly continuous semigroup $\exp\{-tA\}$.

As is known, for sufficiently smooth v_0 and $f(t)$, problem (1) has a unique continuously differentiable solution $v(t)$, and

$$v(t) = \exp\{-tA\}v_0 + \int_0^t \exp\{-(t-s)A\}f(s)\,ds. \tag{2}$$

In the theory of semigroups it is proved that the operator $A + \lambda I$ has a bounded inverse if $\operatorname{Re}\lambda \geqslant \sigma_0$ and σ_0 is a sufficiently large positive number. If

$$\|(A + \lambda I)^{-1}\|_{E\to E} \leqslant C(|\lambda| + 1)^{-1} \qquad (\operatorname{Re}\lambda \geqslant \sigma_0), \tag{3}$$

then we shall call A strongly positive. It is known that in this case the semigroup is continuously differentiable for $t > 0$, and

$$\|A\exp\{-tA\}\|_{E\to E} \leqslant C(T)t^{-1} \qquad (0 < t \leqslant T). \tag{4}$$

2. The totality of all continuous on $[0, T]$ functions $w(t)$ with values in E, satisfying, for some $\alpha \in (0, 1)$, the condition

$$t^{\alpha}(\Delta t)^{-\alpha}\,\|w(t + \Delta t) - w(t)\|_E \leqslant C(\alpha) \qquad (0 < t < t + \Delta t \leqslant T),$$

will be denoted by $C_0^{\alpha}(T)$. This is a Banach space with norm

$$\|w\|_{C_0^\alpha(T)} = \max_{0\leqslant t\leqslant T}\|w(t)\|_E + \sup_{0<t<t+\Delta t\leqslant T} t^\alpha(\Delta t)^{-\alpha}\|w(t+\Delta t)-w(t)\|_E. \tag{5}$$

If, for every $v_0 \in D(A)$ and every $f(t) \in C_0^\alpha(T)$, there exists such a solution of problem (1) that

$$\left\|\frac{dv}{dt}\right\|_{C_0^\alpha(T)} + \|Av\|_{C_0^\alpha(T)} \leqslant C(\alpha,T)(\|f\|_{C_0^\alpha(T)} + \|Av_0\|_E), \tag{6}$$

then we shall say that, for problem (1), coercivity holds in $C_0^\alpha(T)$.

Theorem 1. *In order that coercivity hold for problem (1) in $C_0^\alpha(T)$, it is necessary and sufficient that the operator A be strongly positive in E.*

Necessity. Let f_0 be an arbitrary element of E, and let v_0 be the solution of the equation $\lambda v_0 + Av_0 = f_0$, which exists for $\operatorname{Re}\lambda \geqslant \sigma_0$. The function $v(t) = \exp\{t\lambda\}v_0$ is a solution of problem (1) for $f(t) = \exp\{t\lambda\}f_0$. Using (6), we obtain (3).

To prove **sufficiency**, one must use estimate (4) and apply the ordinary theory of singular integrals to the singular integral

$$A\int_0^t \exp\{-(t-s)A\}f(s)\,ds \tag{7}$$

The inequality (6) makes it possible to study a problem more general than (1),

$$\frac{dv}{dt}A(t)v = f(t), \qquad v(0) = v_0, \tag{8}$$

where $A(t)$, for each $t \in [0,T]$, is a strongly positive operator whose domain of definition does not depend on t. Let $A(0) = A$, and let $A + \sigma_0 I$ have an inverse. Introduce the operator $B(t) = A(t)(A+\sigma_0 I)^{-1}$ and, for example, suppose that the operator-valued function $B(t)$ satisfies on $[0,T]$ a Hölder condition with exponent α. Then for solutions of problem (8) coercivity holds in $C_0^\alpha(T)$.

3. In the theory of problem (1) on the finite segment $[0,T]$, without loss of generality one may assume that $\sigma_0 = 0$. Then for any $\alpha \in [0,1]$ and, for example, for any $v_0 \in D(A)$, the number

$$|v_0|_\alpha = \left(\int_0^\infty \|A\exp\{-tA\}v_0\|_E^{\frac{1}{1-\alpha}}\,dt\right) \tag{9}$$

is finite.

The set of all those v_0 for which $|v_0|_\alpha < \infty$ forms a linear set E_α. If A is strongly positive, then E_α is a Banach space with norm (9). In ([1]) fractional powers A^α of a broad class of operators A, acting in arbitrary Banach spaces, were introduced. Such operators include, in particular, strongly positive operators. The domain of definition D_α of the operator A^α is a Banach space with norm $\|v_0\|_\alpha = \|A^\alpha v_0\|_E$.

Theorem 2. *Let A be strongly positive, $\alpha \in (0,1)$, and $\varepsilon \in (0,\alpha)$. Then:*

$$\text{if } v_0 \in D_\alpha, \quad \text{then } v_0 \in E_{\alpha-\varepsilon} \text{ and } |v_0|_{\alpha-\varepsilon} \leqslant C(\alpha,\varepsilon)\|v_0\|_\alpha;$$

$$\text{if } v_0 \in E_\alpha, \quad \text{then } v_0 \in D_{\alpha-\varepsilon} \text{ and } \|v_0\|_{\alpha-\varepsilon} \leqslant C(\alpha,\varepsilon)|v_0|_\alpha.$$

The question arises for which spaces E and operators A Theorem 2 remains valid also for $\varepsilon = 0$.

If E is a Hilbert space H, and A is a positive definite self-adjoint operator in it, then for $\alpha \geqslant 1/2$, if $v_0 \in D_\alpha$, then $v_0 \in H_\alpha$ and $|v_0|_\alpha \leqslant C(\alpha)\|v_0\|_\alpha$. M. A. Krasnosel' skii showed that for $\alpha \in (0,1/2)$ the last assertion is, generally speaking, false. Finally, it is easy to see that $D_{1/2} = H_{1/2}$.

4. By $B_p(T)$ $(p \geqslant 1)$ we denote the Bochner space of strongly measurable functions $w(t)$ on $[0,T]$ with values in E, whose norm is summable to the power p. The norm in $B_p(T)$ is defined, as is known, by the formula

$$\|w\|_p^T = \left(\int_0^T \|w(t)\|_E^p \, dt\right)^{1/p}. \tag{10}$$

If for any $f(t) \in B_p(T)$ and any $v_0 \in E_{1/q}$, where $1/p+1/q=1$, there exists such an absolutely continuous solution $v(t)$ of problem (1) that $dv/dt, Av \in B_p(T)$, and the function $v(t)$ is continuous on $[0,T]$ in the norm of the space

$E_{1/q}$, and for every $t \in [0,T]$ the inequality

$$\left\|\frac{d\bar{v}}{dt}\right\|_{-p} + \|A\bar{v}\|_p^t + |v(t)|_{1/q} \leq C(p,T)\left(\|f\|_{-p}^t + |v_0|_{1/q}\right), \tag{11}$$

holds, then we shall say that coercivity holds for problem (1) in $B_p(T)$.

Theorem 3*. *In order that coercivity in $B_p(T)$ hold for problem (1), it is necessary that the operator A be strongly positive in E. This condition is also sufficient if $1 < p < \infty$ and if*

$$\text{coercivity holds for some } r \in (1,\infty). \tag{*}$$

The necessity is proved in the same way as in Theorem 1. The proof of sufficiency is based on a theorem from the theory of singular integrals in $B_p(T)$ ([2]) and on the well-known Hardy inequality.

If E is H, then with the aid of the Fourier transform it is shown that coercivity holds in $B_2(T)$. Therefore, by Theorem 3, coercivity holds also in any $B_p(T)$.

Inequality (11) makes it possible to show that coercivity in $B_p(T)$ also holds for the solutions of problem (8), if $B(t)$ (see item 2) has only discontinuities of the first kind.

5. Let Ω be a bounded domain of n-dimensional space with a sufficiently smooth boundary. In the cylinder $Q = \Omega \times [0, T]$ consider the parabolic system

$$\frac{\partial \bar{v}}{\partial t} + A\left(x, \frac{\partial}{\partial x}\right) \bar{v} = \bar{f}(t, x) \tag{12}$$

with an elliptic differential operator in partial derivatives of order $2l$. For solutions of system (12) satisfying homogeneous boundary conditions, for example in ([3]), coercivity inequalities have been established in Schauder norms and in the norms $L_p(Q)$. In these inequalities the norms with respect to t and with respect to the spatial variables are the same.

The boundary-value problem for system (12) can be regarded as the Cauchy problem (1) in various functional spaces. The application of Theorems 1 and 3 to the results of ([3]) makes it possible to obtain a series of new coercivity inequalities with different norms with respect to t and with respect to the spatial variables. For lack of space we do not write out these rather cumbersome inequalities.

In ([4]), and then also in ([3]), the first boundary-value problem for the linearized Navier–Stokes system in the cylinder Q was considered, and coercivity inequalities were established for the solutions of this system. Theorems 1 and 3 here also lead to new coercivity inequalities.

A second consequence of Theorems 1 and 3 for parabolic systems and the Navier–Stokes system is that the corresponding stationary systems generate strongly positive operators. This, in particular, makes it possible to develop an L_p-theory for the nonlinear nonstationary Navier–Stokes system and to prove that in the n-dimensional case a local theorem of existence and uniqueness of the solution of the first boundary-value problem for such a system is valid for any solenoidal initial velocity v_0 from $L_n(\Omega)$.

6. Coercivity inequalities also hold in norms with derivatives. We formulate here only one result.

Let $w(t)$ be a function continuous on $[0, T]$ with values in E. Suppose that for $t > 0$ this function is k times continuously differentiable and its derivative $w^{(k)}(t)$ satisfies a Hölder condition with exponent α. Finally, suppose that

$$\sum_{i=0}^{k} \sup_{0<t\leq T} t^i \|w^{(i)}(t)\|_E + \sup_{0<t<t+\Delta t\leq T} t^{k+\alpha}(\Delta t)^{-\alpha}\|w^{(k)}(t+\Delta t) - w^{(k)}(t)\|_E < \infty. \tag{13}$$

* Apparently, Theorem 3 is also true without condition (*).

The totality of all such functions forms a Banach space $C_0^{\alpha+k}(T)$ with norm $\|w\|_{C_0^{\alpha+k}(T)}$, defined by the left-hand side of inequality (13).

We shall say that coercivity in $C_0^{\alpha+k}(T)$ holds for the solutions of problem (1) if, for any $v_0 \in D(A)$ and $f(t) \in C_0^{\alpha+k}(T)$, there exists a solution of problem (1) such that $dv/dt,\ Av \in C_0^{\alpha+k}(T)$, and the inequality

$$\left\|\frac{dv}{dt}\right\|_{C_0^{\alpha+k}(T)} + \|Av\|_{C_0^{\alpha+k}(T)} \leq C(\alpha+k, T)\left(\|f\|_{C_0^{\alpha+k}(T)} + \|Av_0\|_E\right) \tag{14}$$

holds.

A generalization of Theorem 1 is

Theorem 4. *In order that coercivity in $C_0^{\alpha+k}(T)$ hold for the solutions of (1), it is necessary and sufficient that the operator A be strongly positive in E.*

This theorem makes it possible to investigate problem (8) and to show that coercivity in $C_0^{\alpha+k}(T)$ holds for its solutions if, for example, the operator-function $B(t)$ is k times continuously differentiable on $[0, T]$ and its derivative $B^{(k)}(t)$ satisfies a Hölder condition with exponent α.

In conclusion we merely note that analogous assertions also hold in the spaces $B_p(T)$.

Received
4 II 1964

REFERENCES

[1] M. A. Krasnosel' skii, P. E. Sobolevskii, DAN, **129**, No. 3 (1959).
[2] A. Benedek, A. P. Calderon, R. Panzone, Proc. Nat. Acad. Sci. USA, **48**, No. 3 (1962).
[3] V. A. Solonnikov, Reports at the joint Soviet-American symposium, Novosibirsk, 1963.
[4] V. I. Yudovich, DAN, **130**, 1214 (1960).

Note: Figure translations are in progress. See original paper for figures.