



Soviet-era science, translated into English

MATHEMATICS

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.40955>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

A. R. ESAYAN, V. Ya. STETSKENKO

ESTIMATES OF THE SPECTRUM OF INTEGRAL OPERATORS AND INFINITE MATRICES

(Presented by Academician A. Yu. Ishlinskii on 20 II 1964)

Let E be a real Banach space, and let A be a linear bounded operator acting in E . If the spectrum of the operator A is contained in the disk $|\lambda| \leq r$, then, as is known ¹, for every $f \in E$ ($f \neq 0$) the equation $\lambda x = Ax + f$, for $|\lambda| > r$, has a unique solution x^* , to which the method of successive approximations converges,

$$x_{n+1} = \frac{1}{\lambda}(Ax_n + f) \quad (n = 0, 1, \dots),$$

starting from an arbitrary element $x_0 \in E$. In connection with this, it is of interest to estimate the spectrum of a linear bounded operator A . Such an estimate is given, for example, by the norm $\|A\|$ of the operator A . The exact value of the spectral radius $r(A)$ of the operator A can be computed by the formula ¹

$$r(A) = \lim_{n \rightarrow \infty} \sqrt[n]{\|A^n\|}. \tag{1}$$

Estimating the spectrum of an operator by means of its norm is, as a rule, crude, while determining the spectral radius by formula (1) is very difficult.

In the present article a method is proposed for estimating the spectral radius of linear integral operators acting in the spaces $C(\Omega)$, $L_p(\Omega)$ ($p > 1$), and also the spectral radius of infinite matrices generating linear bounded operators acting in one or another space of sequences. This method is based on the following theorem ².

Theorem 1. *Let the cone K be normal and reproducing ^(3,4), the linear operator A be positive, u_0 -bounded above, and*

$$A^m u_0 \leq \lambda_0 u_0 \quad (u_0 \in K).$$

Let, further, the linear operator A_0 satisfy the condition

$$-A \leq A_0 \leq A.$$

Then the spectrum of the operator A_0 lies in the disk

$$|\lambda| \leq \sqrt[m]{\lambda_0}.$$

1. Let $E = C(\Omega)$, where Ω is a closed bounded set of a finite-dimensional space. Let the function $K(t, s)$ ($t, s \in \Omega$) be such that the operators

$$A_0 x(t) = \int_{\Omega} K(t, s)x(s) ds, \quad Ax(t) = \int_{\Omega} |K(t, s)|x(s) ds \quad (2)$$

act in $C(\Omega)$ and are bounded. We distinguish in $C(\Omega)$ the cone K of nonnegative functions. This cone is normal and solid. Put

$$u_0(t) = \int_{\Omega} |K(t, s)| ds.$$

The operator A is $u_0(t)$ -bounded above, since for every $x(t) \in K$

$$Ax(t) = \int_{\Omega} |K(t, s)|x(s) ds \leq \max_{s \in \Omega} |x(s)| u_0(t) = \|x\|u_0. \quad (3)$$

From Theorem 1 it then follows that

Theorem 2. Let λ_0 be a real number such that

$$\int_{\Omega} |K(t, s)| \left[\lambda_0 - \int_{\Omega} |K(s, \tau)| d\tau \right] ds \geq 0. \quad (4)$$

Then the spectral radius of the operator A_0 does not exceed λ_0 :

$$r(A_0) \leq \lambda_0.$$

Remark. Relation (4) is always satisfied if for λ_0 one takes the quantity

$$\lambda_0 = \|A_0\| = \max_s \int_{\Omega} |K(s, \tau)| d\tau.$$

However, as a rule, λ_0 can be chosen considerably smaller.

A generalization of Theorem 2 is

Theorem 3. Let

$$R_m(t, s) = \int_{\Omega} \cdots \int_{\Omega} |K(t, s_1)| \cdot |K(s_1, s_2)| \cdots |K(s_{m-1}, s)| ds_1 \cdots ds_{m-1},$$

where m is an arbitrary positive integer, and let λ_1 be such that

$$\int_{\Omega} |K(t, s)| \cdot \left[\lambda_1^m - \int_{\Omega} R_m(s, \tau) d\tau \right] ds \geq 0. \quad (5)$$

Then the estimate holds

$$r(A_0) \leq \lambda_1.$$

Remark. Inequality (5) is satisfied for

$$\lambda_1 = \left[\max_t \int_{\Omega} R_m(t, s) ds \right]^{1/m}.$$

If one takes a positive function from $C(\Omega)$ such that

$$\min_{t \in \Omega} \psi(t) = \delta > 0,$$

then the operator A will be $\vartheta_0(t)$ -bounded above, where

$$\vartheta_0(t) = \psi(t) \int_{\Omega} |K(t, s)| ds = \psi(t) u_0(t).$$

Indeed, for any $x(t) \in K$,

$$\int_{\Omega} |K(t, s)| x(s) ds \leq \|x\|_{C(\Omega)} u_0(t) \leq \frac{\|x\|}{\delta} u_0(t) \psi(t) = a(x) \vartheta_0(t)$$

$$\left(a(x) = \frac{\|x\|}{\delta} \right).$$

Therefore $r(A_0) \leq \lambda_{\psi}$, where λ_{ψ} is such that

$$\int_{\Omega} |K(t, s)| \left[\lambda_{\psi}^m \psi(t) - \psi(s) \int_{\Omega} R_m(s, \tau) d\tau \right] ds \geq 0.$$

Theorem 4. Let λ_2 satisfy the inequality

$$\int_{\Omega} R_m(t, s) \left[\lambda_2 - \int_{\Omega} |K(s, \tau)| d\tau \right] ds \geq 0.$$

Then

$$r(A_0) \leq \lambda_2.$$

For the proof it is enough to note that the operator A is $g_m(t)$ -bounded from above, where

$$g_m(t) = \int_{\Omega} R_m(t, s) ds.$$

Indeed, for any $x(t) \in K$,

$$\int_{\Omega} R_m(t, s)x(s) ds \leq \|x\|_C g_m(t).$$

Let us note that the same arguments allow one to assert that $r(A_0) \leq \lambda_{\psi}$, where $\lambda_{\psi}(n, m)$ satisfies the condition

$$\int_{\Omega} R_m(t, s) \left[\lambda_{\psi}^n \psi(t) - \psi(s) \int_{\Omega} R_n(s, \tau) d\tau \right] ds \geq 0$$

(n is an arbitrary natural number). We note that $\lambda_{\psi}(n, m)$ is monotone in n, m and $\lim_{n \rightarrow \infty} \lambda_{\psi}(n, m) = r(A_0)$.

2. Let now the space $E = L_p(\Omega)$ be semiordered by the cone K of nonnegative functions. This cone is normal and reproducing. Let the operators A_0 and A act in $L_p(\Omega)$. It is clear that $AK \subset K$.

Theorem 5. *Let*

$$u_0(t) \equiv \left(\int_{\Omega} |K(t, s)|^q ds \right)^{1/q} \in L_p(\Omega) \quad \left(\frac{1}{p} + \frac{1}{q} = 1 \right).$$

Then $r(A_0) \leq \mu_0$, where μ_0 is such that

$$\int_{\Omega} |K(t, s)| \left(\int_{\Omega} |K(s, \tau)|^q d\tau \right)^{1/q} ds \leq \mu_0 \left(\int_{\Omega} |K(t, s)| ds \right)^{1/q}.$$

As μ_0 in this relation one may take the quantity

$$\mu_0 = \left[\int_{\Omega} \left(\int_{\Omega} |K(t, s)|^q ds \right)^{p/q} dt \right]^{1/p} \equiv C,$$

however, in concrete cases μ_0 can be chosen in a better way, i.e. $\mu_0 < C$.

We note that μ_0 can also be chosen from the relation

$$\int_{\Omega} R_n(t, s) \left[\int_{\Omega} R_m^q(s, \tau) d\tau \right]^{1/q} ds \leq \lambda_0^n \left(\int_{\Omega} R_m^q(t, s) ds \right)^{1/q}.$$

Here it is assumed that

$$u_m(t) \equiv \left(\int_{\Omega} R_m^q(t, s) ds \right)^{1/q} \in L_p(\Omega).$$

3. We turn to estimating the spectral radius of an infinite matrix $A_0 = (a_{ij})$ ($i, j = 1, 2, \dots$). Let m be the space of bounded numerical sequences. Put $A = (|a_{ij}|)$.

The infinite matrices A_0, A generate linear bounded operators acting in m if and only if

$$\sum_{k=1}^{\infty} |a_{ik}| \leq M < +\infty.$$

We construct the element

$$\vartheta_0 = \left(\sum_{k=1}^{\infty} |a_{1k}|, \sum_{k=1}^{\infty} |a_{2k}|, \dots, \sum_{k=1}^{\infty} |a_{ik}|, \dots \right).$$

Obviously, $\vartheta_0 \in m$, and the operator A is ϑ_0 -bounded above; moreover,

$$A\vartheta_0 \leq \sup_{i=1,2,\dots} \frac{\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{ij}| \cdot |a_{jk}|}{\sum_{j=1}^{\infty} |a_{ij}|} \vartheta_0.$$

In view of what has been said, the following holds.

Theorem 6. Let λ_0 satisfy the condition

$$\sum_{j=1}^{\infty} |a_{ij}| \left[\lambda_0 - \sum_{k=1}^{\infty} |a_{jk}| \right] \geq 0 \quad (i = 1, 2, \dots).$$

Then

$$r(A_0) \leq \lambda_0.$$

Theorem 6 admits a generalization in the sense of Theorems 3 and 4.

4. Suppose that the matrix $A = (|a_{ij}|)$ satisfies the Riesz condition

$$\sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{ik}|^{p/(p-1)} \right)^{p-1} < \infty.$$

Then the matrix A_0 generates a linear completely continuous operator acting in the space l_p .

Theorem 7. The estimate

$$r(A_0) \leq \sup_{i=1,2,\dots} \frac{\sum_{j=1}^{\infty} |a_{ij}| \left(\sum_{k=1}^{\infty} |a_{jk}|^q \right)^{1/q}}{\left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{1/q}}.$$

is valid.

One may also consider the operator A_0 acting in other Banach spaces semi-ordered by normal and reproducing cones, and, with the aid of Theorem 1, obtain various kinds of estimates for the spectrum of this operator. Moreover, one may consider an operator A_0 such that $A_0 E_1 \rightarrow E_2$, where E_1 and E_2 are Banach spaces semi-ordered respectively by cones K_1 and K_2 , the first of which is reproducing and the second normal.

We note that the results presented remain valid also in the case when a complex Banach space E is considered.

In conclusion, the authors express their gratitude to M. A. Krasnosel'skii for his attention.

Tajik State University

Received
5 II 1964

CITED LITERATURE

1. L. V. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, Moscow, 1959.
2. V. Ya. Stetsenko, *DAN*, **157**, No. 5 (1964).
3. M. G. Krein, M. A. Rutman, *UMN*, **3**, issue 1 (23) (1948).
4. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.
5. Fr. Riesz, *Leçons sur les systèmes d' équations linéaires à une infinité d' inconnues*, Paris, 1913.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.