Kh. Sh. Mukhtarov

In the present article we study the equation

1964

Abstract

Full Text

Kh. Sh. Mukhtarov

Investigation of a Class of Nonlinear Singular Equations with Hilbert Kernel

(Presented by Academician I. N. Vekua, 11 VII 1963)

In the present article we study the equation

$$u(x) = \lambda \frac{a(x)}{2\pi} \int_{-\pi}^{+\pi} f(s, u(s)) \operatorname{ctg} \frac{s-x}{2}\, ds + b(x) \tag{1}$$

in the space $H_k(\varphi)$, which is a generalization of the Hölder space $H_{k,\delta}$.

Let $\varphi(\sigma)$ be a continuous monotonically increasing function satisfying the condition: there exists a constant $c > 1$ such that

$$1 < \varliminf_{\sigma \to 0} \frac{\varphi(c\sigma)}{\varphi(\sigma)} \leq \varlimsup_{\sigma \to 0} \frac{\varphi(c\sigma)}{\varphi(\sigma)} < c, \qquad \varphi(0) = 0.$$

Definition. A periodic function $u(x)$ of period 2π, defined on $[-\pi, \pi]$, is said to **belong to the space** $H_k(\varphi)$ if it satisfies the conditions:

$$|u(x)| \leq k, \qquad |u(x + \Delta x) - u(x)| \leq k\varphi(|\Delta x|),$$

where $k = \text{const}$.

We note that an equation of the form (1) was studied in papers ([1,2]) in the space $H_{k,\delta}$. It is not hard to see that $H_{k,\delta} \in H_k(\varphi)$, and $H_k(\varphi)$ is a complete compact metric space in the sense of the metric $C(-\pi, \pi)$.

In $H_k(\varphi)$ we introduce two metrics. If $u(x), v(x) \in H_k(\varphi)$, then

$$\rho_{C(-\pi,\pi)}(u, v) = \max |u(x) - v(x)|, \tag{2}$$

$$\rho_{L_2(-\pi,\pi)}(u, v) = \left[\int_{-\pi}^{\pi} |u(s) - v(s)|^2\, ds \right]^{1/2}. \tag{3}$$

Since

$$\rho_{L_2(-\pi,\pi)}(u, v) \leq \sqrt{2\pi}\, \rho_{C(-\pi,\pi)}(u, v),$$

then, by virtue of Lemma 3 ([3]), $H_k(\varphi)$ is complete with respect to the metric (3).

We first give several lemmas.

Lemma 1. *If a sequence $\{u_n(x)\} \in H_k(\varphi)$ converges to $u_0(x) \in H_k(\varphi)$ in the sense of the metric (3), then the same sequence also converges in the sense of the metric (2).*

Proof. Let

$$a_n = \int_{-\pi}^{\pi} |u_n(x) - u_0(x)|^2 \, dx, \qquad \lim_{n\to\infty} a_n = 0.$$

Without loss of generality, let $a_n < \pi^2$. Introduce the following sets:

$$E_n(x_0) = \begin{cases} [x_0,\, x_0 + \sqrt{a_n}], & \text{if } -\pi \le x_0 \le 0, \\ [x_0 - \sqrt{a_n},\, x_0], & \text{if } 0 < x_0 \le \pi. \end{cases}$$

Obviously, $E_n(x_0) \subset [-\pi, \pi]$.

Applying the theorem on the mean value of an integral, we obtain

$$a_n \ge \int_{E_n(x_0)} |u_n(x) - u_0(x)|^2 \, dx = |u_n(\xi_n) - u_0(\xi_n)|^2 \operatorname{mes} E_n(x_0).$$

Hence it follows that

$$|u_n(\xi_n) - u_0(\xi_n)| \le \sqrt[4]{a_n}.$$

Since $\xi_n \in E_n(x_0)$, we have $|\xi_n - x_0| \le \sqrt{a_n}$ and, consequently, $\varphi(|\xi_n - x_0|) \le \varphi(\sqrt{a_n})$.

Taking the above into account, we have

$$|u_n(x_0) - u_0(x_0)| \le$$

$$\le |u_n(x_0) - u_n(\xi_n)| + |u_n(\xi_n) - u_0(\xi_n)| + |u_0(x_0) - u_0(\xi_n)| \le$$

$$\le 2k\varphi(|\xi_n - x_0|) + \sqrt[4]{a_n} \le 2k\varphi(\sqrt{a_n}) + \sqrt[4]{a_n}.$$

Thus, for any $x \in [-\pi, \pi]$ we have

$$|u_n(x) - u_0(x)| \leq 2k\varphi(\sqrt{a_n}) + \sqrt[4]{a_n}. \tag{4}$$

If in inequality (4) we pass to the limit as $n \to \infty$ and take into account that $\varphi(0) = 0$, we obtain the assertion of Lemma 1.

Lemma 2. If $u(x) \in H_k(\varphi)$, then

$$V(x) = -\frac{1}{2\pi} \int_{-\pi}^{+\pi} u(s) \operatorname{ctg} \frac{s-x}{2} \, ds$$

satisfies the conditions

$$|V(x)| \leq lk, \qquad |V(x + \Delta x) - V(x)| \leq lk\varphi(|\Delta x|),$$

where l is a constant independent of k.

The assertion of Lemma 2 follows from work (4).

Lemma 3. If $f(s, u)$ is periodic in s with period 2π, defined for $-\pi \leq s \leq \pi$, $-k \leq u \leq k$, and satisfies the conditions

$$|f(s_1, u_1) - f(s_2, u_2)| \leq l_1\varphi(|s_1 - s_2|) + l_2|u_1 - u_2|, \tag{5}$$

$$f(s, 0) \in H_{k_0}(\varphi) \tag{6}$$

and if $a(x) \in H^*_{l_3}(\varphi)$, $b(x) \in H_{k_1}(\varphi)$ $(k_1 < k)$, then the operator

$$Bu = \frac{\lambda a(x)}{2\pi} \int_{-\pi}^{+\pi} f(s, u(s)) \operatorname{ctg} \frac{s-x}{2} \, ds + b(x)$$

for

$$|\lambda| < \lambda_0 = \frac{k - k_1}{2ll_3(2l_1 + l_2k + k_0)}$$

acts from $H_k(\varphi)$ into $H_k(\varphi)$ and satisfies the Lipschitz condition, i.e.

$$\rho_{L_2(-\pi,\pi)}(Bu, Bv) \leq |\lambda| \, l_3 l_2 \, \rho_{L_2(-\pi,\pi)}(u, v).$$

Hence it follows

Lemma 4. Under the conditions of Lemma 3, if

$$|\lambda| < \min\left\{\lambda_0, \frac{1}{l_3 l_2}\right\},$$

equation (1) has a unique solution $u_0(x) \in H_k(\varphi)$, whose successive approximations converge in the sense of the metric (3).

If Lemma 1 is taken into account, then these successive approximations converge in the sense of the metric (2).

Thus, we have proved

Theorem. If $f(s, u)$ satisfies conditions (5), (6), and $a(x) \in H_{l_3}(\varphi)$, $b(x) \in H_{k_1}(\varphi)$ $(k_1 < k)$, then the nonlinear singular integral equation (1) has a unique solution in $H_k(\varphi)$ (for small $|\lambda|$), and this solution can be found by the method of successive approximations; the successive approximations will converge uniformly.

Dagestan State University
named after V. I. Lenin

Received
6 VII 1963

REFERENCES

1. A. I. Guseinov, Mat. sborn., **20** (62), No. 2 (1947).
2. A. A. Babaev, Uch. zap. ASU named after S. M. Kirov, ser. phys.-math. and chem. sciences, No. 4 (1961).
3. A. A. Babaev, Uch. zap. ASU named after S. M. Kirov, ser. phys.-math. and chem. sciences, No. 2 (1960).
4. N. K. Bari, Izv. AN SSSR, ser. matem., **19**, 285 (1955).

Note: Figure translations are in progress. See original paper for figures.