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Abstract

Full Text

MATHEMATICS

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A STATISTICAL APPROACH TO THE WELL-POSEDNESS OF PROBLEMS OF MATHEMATICAL PHYSICS

(Presented by Academician V. I. Smirnov on 28 IX 1963)

Since the initial and boundary values in problems of mathematical physics are usually determined, in real formulations, from experimental data (and, consequently, with inevitable errors), the natural question arises of the well-posedness (in the sense of Hadamard (1)) of the problems under consideration, i.e., of the continuous dependence of the solution on the initial and boundary conditions. Here, of course, an essential role is played by the topologization of the spaces of solutions and of the initial and boundary data. As is known, many problems of mathematical physics turn out to be well posed; however, Hadamard's example shows the existence of problems that are ill posed and nevertheless (see the literature in (2)) have a perfectly clear physical meaning and practical application. In this connection, the study of ill-posed problems has recently attracted the attention of many authors (2-8). The aim of these studies is in fact such a reasonable modification of the original Hadamard definition of well-posedness that, in the sense of the new definition, problems from a broader class would turn out to be well posed.

For this purpose, in the works of the cited authors one usually restricts in advance the set of solutions by some prescribed compact set (for example, in the case of the Cauchy problem for the Laplace equation, by the set of harmonic functions bounded by a fixed constant). The operator assigning to initial and boundary data the solution then obviously realizes a homeomorphism of this compact set and the corresponding set of initial and boundary conditions. An a priori estimate of the solution has made it possible in a number of cases to indicate an effective algorithm for constructing the solution and to give a proof of well-posedness (in the refined sense) of the problems under study.

A different circle of ideas includes the statistical approach to the study of the well-posedness of problem formulations, connected with the assumption that the initial and boundary values obtained from experimental data are realizations of random processes (fields). Let us begin the consideration with the already mentioned example of Hadamard.

As is known, the solution of the Cauchy problem for the Laplace equation in

the upper half-plane with initial conditions

$$u_n|_{y=0} = 0, \quad \frac{du_n}{dy}\bigg|_{y=0} = e^{-\sqrt{n}} \cos nx \quad (1)$$

is unbounded for any $y > 0$, when $n \rightarrow \infty$, although the Cauchy data in this case tend uniformly to zero together with all derivatives. If, however, the parameter n is regarded as random, then one may speak of the distribution of the values assumed by the solution at a fixed point and, for example, compare the variances of the distributions of n and u_n . This model example suggests studying the distribution of various functionals of the solution

of the Cauchy problem as a function of the parameters of the random process whose realizations we take as the Cauchy data. Let us note that, in doing so, it is entirely unnecessary, as in the deterministic approach, to specify a particular metric in the spaces of initial and boundary conditions and of solutions.

For definiteness we shall speak of the two-dimensional Cauchy problem for the Laplace equation in a bounded simply connected domain with piecewise smooth boundary. By means of a conformal mapping and the solution of the Dirichlet problem (the study of which is carried out without difficulty with the aid of the maximum principle), this problem reduces to finding a 2π -periodic in x harmonic function $u(x, y)$ in the strip $0 < y < l$, from the prescribed values

$$u(x, 0) = 0, \quad u'_y(x, 0) = \varphi(x). \quad (2)$$

We regard $\varphi(x)$ as the Cauchy data of a "disturbance," which is superposed (the problem being linear) on the solution of the Cauchy problem with certain "true" initial conditions. We are interested in the following.

- a) What should the random process be so that its realizations $\varphi(x)$ can serve as Cauchy data and so that, for some $l > 0$, the mathematical expectation $E(\|u(x, l)\|_{L^2[0, 2\pi]}^2)$ is finite?
- b) In what way must a sequence of random processes φ_n tend to the degenerate one (with probability 1, $\varphi(x) \equiv 0$) in order that

$$E(\|u_n(x, l)\|^2) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3)$$

The answer to these questions is given by the following

Theorem. Let $\varphi(x)$ be a realization of a 2π -periodic random process with correlation function $r(s, t)$, and let $u(x, y)$ be a 2π -periodic in x harmonic function in the strip $|y| < l$, with $u(x, 0) = 0$, $u'_y(x, 0) = \varphi(x)$. Let $\{\lambda_k\} = \{\sigma_k^2\}$

and $\{\psi_k(x)\}$ be the eigenvalues and normalized eigenfunctions of the integral operator

$$K_\psi = \int_0^{2\pi} r(s, t)\psi(t) dt.$$

Then

$$\begin{aligned} E(\|u(x, l)\|_{L^2[0, 2\pi]}^2) &= \\ &= \frac{1}{\pi} \left(\frac{1}{2} l^2 \sum_{i=1}^{\infty} \lambda_i (1, \psi_i)^2 + \sum_{k=1}^{\infty} \frac{\text{sh}^2 kl}{k^2} \sum_{i=1}^{\infty} \lambda_i ((\cos kx, \psi_i)^2 + (\sin kx, \psi_i)^2) \right). \end{aligned} \quad (4)$$

This assertion is to be understood in the sense that if the series on the right-hand side of the equality converges, then the functional $\|u(x, l)\|^2$ is defined for almost all realizations $\varphi(x)$, has a mathematical expectation, and equality (4) holds. In particular, in order that the mathematical expectation $\|u_n(x, l)\|^2$ tend to zero, it is necessary and sufficient that, as $n \rightarrow \infty$, the quantities

$$\sum_{i,k} \left(\left(\frac{e^{kl} \cos kx}{k}, \sigma_i^{(n)} \psi_i^{(n)} \right)^2 + \left(\frac{e^{kl} \sin kx}{k}, \sigma_i^{(n)} \psi_i^{(n)} \right)^2 \right), \quad (5)$$

tend to zero, where $\{\sigma_i^{(n)2}\}$ and $\{\psi_i^{(n)}\}$ are the eigenvalues and eigenfunctions of the integral operator with kernel $r_n(s, t)$. If the process φ is stationary and $r(s, t) = r(s - t)$, where

$$r(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dF(\lambda) = \sum_{k=0}^{\infty} p_k \cos kt, \quad (6)$$

the preceding conditions (4), (5) are rewritten in the form

$$E(\|u(x, l)\|^2) = p_0 l^2 + 2 \sum_{k=1}^{\infty} \frac{p_k \text{sh}^2 kl}{k^2}; \quad (7)$$

$$\sum_{k=0}^{\infty} \frac{p_k^{(n)} e^{2kl}}{k^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (8)$$

Thus the concept of statistical correctness is introduced (in our case, for the Cauchy problem for the Laplace equation) with respect to initial data that are realizations of a random process from some class of processes with convergence to the expressed quantity defined in it in accordance with (3).

With the aid of the theorem presented above one can obtain an answer to the questions posed in a) and b) also in the case of an arbitrary elliptic equation with coefficients periodic in x (with the same Cauchy data (2)), which, as is known, is reduced by a change of variables to the Laplace equation.

Questions a) and b) can, of course, be posed not only with respect to problems ill-posed in Hadamard's sense. If, instead of the Laplace equation, we considered the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (9)$$

with the same Cauchy data (2), then it would turn out that (in the case of a stationary process) the condition analogous to (7) takes the form

$$E (\|u(x, l)\|^2) = l^2 p_0 + 2 \sum_{k=1}^{\infty} \frac{p_k \sin^2 kl}{k^2}, \quad (10)$$

and for the mathematical expectation $\|u(x, l)\|^2$ to tend to zero it is sufficient that

$$\sum_{k=1}^{\infty} \frac{p_k^{(n)}}{k^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (11)$$

From comparison of (7), (8) and (10), (11) it follows that the Cauchy problem for the Laplace equation (ill-posed in Hadamard's sense) is statistically correct with respect to a rather narrow class of random processes, whereas in the case of the Cauchy problem for the wave equation, which is well-posed in Hadamard's sense, statistical correctness holds for a broad class of processes, including even some generalized random processes.

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