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ON THE FOUNDATIONS OF GEOMETRY

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Abstract

Full Text

MATHEMATICS

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ON THE FOUNDATIONS OF GEOMETRY

(Presented by Academician V. I. Smirnov on 14 X 1963)

The aim of this work is to construct a unified axiomatic system for spaces with free mobility that carry a metric which is positive definite, alternating, or degenerate. If kinematics is regarded as the geometry of space–time, then our axiomatic systems encompass cosmological models that proceed from the equality of the points of space–time. In doing so, the axioms are purely kinematic; equations of gravitation are not used.

1°. We begin with plane geometries. The initial objects are: a **point** A and a **line** a. We introduce the relations: A/B (A is close to B); ABC (B is between A and C); $A \in a$ (A lies on a); $A \to A'$ and $a \to a'$ (A is moved into A'), etc. The axioms of order are similar to the corresponding axioms of ([1]), vol. 2, § 82. But axiom I_2, stating that of three distinct points of a line at least one lies between the other two, is formulated only for pairwise close points of a line. The axioms of incidence are similar to the axioms of ([2]), § 2. Only in axioms I_1 and I_2 are the words "for any two close points" introduced. The axioms of structure ([1]) and the version of the axioms of motion used are supplemented analogously.

Among the lines one singles out a class of lines called **postulated**. It is described by the axioms:

1. A postulated line can be moved only into a postulated line, and into any postulated line.
2. A postulated line can be rotated about any of its points in both directions.
3. This rotation can be carried out continuously, without leaving the class of postulated lines.
4. A segment of a postulated line cannot be contracted (moved into its proper part).
5. An angle made up of postulated lines cannot be contracted.

We say that a is **perpendicular** to b **equally with** a_1, if, under some reflection of one of the half-planes determined by the line b into the other, simultaneously $a \to a$, $a_1 \to a_1$. We write this in the form $a, a_1 \perp b$. We say that a **point** A is **close to a line** a ($A \mid a$), if there exists A', close to A and symmetric to A with respect to a.

We introduce coordinates. From a point O on a postulated line x we erect the (existing and unique) $y \perp x$. From M we drop $MX \perp x$ and MY, $x \perp y$. The

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segments OX and OY are called the **coordinates** of the point M. From the formulated axioms of this kind of "absolute geometry" it follows that, for fixed x and O, every point M $(M \mid x,\ M \mid y)$ has uniquely determined coordinates.

2°. In order to define a concrete geometry, one must add to the axioms of absolute geometry two variable postulates, each of which may have three mutually exclusive formulations.

A. We consider a pencil of lines. One of three possibilities is possible.

I_1. Every line of the pencil is postulated (homogeneous plane).

II_1. In the pencil there is a unique non-postulated line (degenerate plane).

III_1. In a pencil there are two or more non-postulated straight lines (a pseudo-plane).

It is proved that for all pencils of straight lines one and the same one of the postulates A is valid. The choice of one of these postulates specifies the possible geometries according to the property of homogeneity of straight lines, or, as we shall say, according to 1-homogeneity.

B. Let us consider a trirectangle with equal perpendiculars. One of three possibilities is possible: the lateral side of the trirectangle is greater than, equal to, or less than the opposite base. This corresponds to the planimetries of Riemann, Euclid, and Lobachevsky. It is proved that for all trirectangles one and the same one of the postulates B holds. We say that the choice of one of the postulates B specifies the possible geometries according to the property of equidistance or, better, 0-homogeneity. Combining, we obtain 9 plane geometries.

3°. By Gauss' s method (([1]), vol. I, pp. 347–357), the equations of the triangle and the trirectangle are found. All 9 trigonometries can be written uniformly as formulas of spherical trigonometry on the sphere

$$(\xi j_1)^2 + (\eta j_1 j_2)^2 + \zeta^2 = 1. \tag{1}$$

Here ξ, η, ζ are real, while j_1, j_2 may represent different types of units $1, e, i$, where $e^2 = 0$.

The radius vector OM is related to the coordinate segments OX, OY by the relations

$$\mathrm{tg}^2 \frac{OM}{R}\, j = \mathrm{tg}^2 \frac{OX}{R_1}\, j_1 + \mathrm{tg}^2 \frac{OY}{R_2}\, j_1 j_2. \tag{2}$$

Here R_1 is the "natural scale" of the postulated straight lines; R_2, of the straight lines perpendicular to the postulated ones; $R = R_1$ or $R = R_2$; $j = j_1$ or $j = j_1 j_2$, depending on the type of straight line; analytic functions of a non-real argument are defined, as usual, by formal expansion in a series (in particular, $\mathrm{tg}\, xe \equiv xe$)*.

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Table 1

Equidistance / 1-homogeneity	Homogeneous plane$j_2 = 1$(does not exist)	Degenerate plane$j_2 =$ e(Newtonian)	Pseudo-plane$j_2 =$ i(Zilbersteinian)
Riemann$j_1 =$ 1(convergence)	[coordinate-net diagram]	[coordinate-net diagram]	[coordinate-net diagram]
Euclid$j_1 =$ e(existence of a rigid body)	[coordinate-net diagram]	[coordinate-net diagram]	[coordinate-net diagram]
Lobachevsky$j_1 =$ i(divergence)	[coordinate-net diagram]	[coordinate-net diagram]	[coordinate-net diagram]

4°. In Table 1 are shown (locally) the coordinate nets for the nine cases. Thin lines depict the postulated straight lines, thick lines the non-postulated ones, and dashed lines the unique non-postulated or limiting non-postulated ones.

5°. Euclidean planes ($j_1 = e$) are modeled by the arithmetic affine plane, where the admissible motions are considered to be parallel

* On the numbers e, see, for example, ([3]), chap. 5.

translation and either an elliptic rotation ($j_2 = 1$), or a shift along a line ($j_2 = e$), or a hyperbolic rotation ($j_2 = i$). The interior of a quadrilateral of a non-Euclidean plane is modeled by a quadrilateral on the sphere (1), where

$$\xi = \zeta \operatorname{tg} \frac{OX}{R_1} j_1, \qquad \eta = \xi \operatorname{tg} \frac{OY}{R_2} j_1 j_2, \qquad \zeta = \cos \frac{OM}{R} j. \tag{3}$$

Therefore, in small finite domains the introduced planimetries are determined uniquely. But in general, many-sheeted coverings of the sphere are possible.

6°. For the construction of n-dimensional geometries, postulated K-planes are singled out. Geometries are classified according to the property of k-homogeneity: in a $(k+1)$-dimensional space one considers a pencil of hyperplanes containing a common postulated $(k-1)$-plane, and formulates one of three postulates, denoted I_k, II_k, III_k, as in the classification by 1-homogeneity. The axiomatics we have constructed makes it possible, uniformly, exhaustively, and in the spirit of Euclid—Hilbert, to obtain all 3^n n-dimensional geometries corresponding to the Cayley—Klein systems ((1), vol. 2, ch. 17).

7°. The 1-inhomogeneous, 2- and 3-homogeneous four-dimensional spaces are given a kinematic interpretation. A point is interpreted as an event, a postulated line as an inertial particle, motion as a transformation of the reference system, and nearness as the possibility of sending a light signal. The choice of the postulate of 1-inhomogeneity III_1 singles out kinematics with a limiting velocity. The postulates of equidistance mean, respectively, that the distance between two

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instantaneously mutually immobile inertial particles increases or decreases with time. The first case gives a cosmology of an "expanding Universe," infinite in time but spherical in space. The second gives the usual special theory of relativity. The third case gives a cosmology of a "pulsating Universe," possibly closed in time*; the space itself is then Lobachevskian.

8°. The preceding results were obtained in 1958–1960. The proposed axiomatics gives a foundation, independent of projective geometry, for semi-Euclidean and semi-non-Euclidean spaces ([5]). These spaces are suitable for the geometric modeling of objects with heterogeneous quantities. In particular, a five-dimensional space with postulates III_1, I_2, I_3, II_4 is interpreted as time—space—electricity—(action) and represents an invariant geometric apparatus for a unified theory of the gravitational and electromagnetic fields.

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[1] V. F. Kagan, *Foundations of Geometry*, Moscow, 1949 and 1956. [2] D. Hilbert, *Foundations of Geometry*, Moscow, 1948. [3] B. A. Rosenfeld, *Non-Euclidean Geometries*, Moscow, 1955. [4] R. I. Pimenov, Tr. III All-Union Mathematical Congress, 4, 1959, p. 78. [5] E. U. Yasinskaya, DAN, 137, No. 6, 1327 (1961).

* In ([4]), p. 79, line 21, instead of "time is closed" one should read: "time may be closed."

Note: Figure translations are in progress. See original paper for figures.

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