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Abstract

Full Text

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ON INTEGRAL TRANSFORMATIONS IN THE COMPLEX DOMAIN

1°. In the author's paper (1) a theory was developed of integral transformations with special kernels formed by means of functions of Mittag-Leffler type

$$E_\rho(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})} \quad (\mu > 0,\ \rho > 0). \tag{1}$$

The concluding result of the theory of such transformations was the construction, in closed form, of an operator of Fourier–Plancherel type for functions summable with square on an arbitrary finite system of rays issuing from the point $z = 0$ of the complex plane.

In another paper by the author, jointly with A. E. Avetisyan (2), a parametric representation was established for the class $M_2(\alpha;\omega)$ $(1/2 < \alpha < \infty,\ -1 < \omega < 1)$ of functions $F(z)$ analytic in the angular domain

$$\Delta(\alpha) : \{|\arg z| < \pi/2\alpha,\ 0 < |z| < \infty\} \tag{2}$$

and satisfying there the condition

$$\sup_{|\varphi|<\pi/2\alpha} \int_0^\infty |F(re^{i\varphi})|^2 r^\omega dr < +\infty. \tag{3}$$

Namely, the following theorem was proved, which is an essential generalization of the well-known Paley–Wiener theorem on the representation of functions of the class H_2 in a half-plane (3):

The class $M_2(\alpha;\omega)$ coincides with the set of functions admitting a representation of the form

$$F(z) = \int_0^\infty E_\rho\left(e^{i\pi/2\gamma} z\tau^{1/\rho};\mu\right)\varphi_1(\tau)\tau^{\mu-1}d\tau + \int_0^\infty E_\rho\left(e^{-i\pi/2\gamma} z\tau^{1/\rho};\mu\right)\varphi_2(\tau)\tau^{\mu-1}d\tau,$$

$$z \in \Delta(\alpha), \tag{4}$$

where $\rho \geq \alpha/(2\alpha-1)$, $1/\gamma = 1/\alpha + 1/\rho$, $\mu = (1+\omega+\rho)/2\rho$, *and the functions* $\varphi_{1,2}(\tau) \in L_2(0,+\infty)$ *are arbitrary.*

Thus, by this theorem the general form of the Paley-Wiener operator was established for an angular domain of arbitrary opening.

In connection with the indicated results the following general problem arises quite naturally:

To construct an operator of Fourier-Plancherel and Paley-Wiener type for any plane sets consisting of a finite number of nonoverlapping rays issuing from the point $z = 0$, and angular domains with vertex at the same point.

In the present note a complete solution of the posed problem is given.

2°. Let us introduce a number of necessary notations and definitions. For arbitrary α $(1/2 < \alpha \leq \infty)$ and ϑ $(-\pi < \vartheta \leq \pi)$, denote by $\Delta(\alpha;\vartheta)$ the set of points coinciding with the angular domain $\{|\operatorname{Arg} z - \vartheta| < \pi/2\alpha,\ 0 < |z| < \infty\}$ for $1/2 < \alpha < \infty$, and with the ray $\{\operatorname{Arg} z = \vartheta,\ 0 < |z| < \infty\}$ for $\alpha = +\infty$. Henceforth we shall assume that the collections of numbers $\{\vartheta_k\}_1^p$: $-\pi < \vartheta_1 < \vartheta_2 < \dots < \vartheta_p \leq \pi$; $\{\alpha_k\}_1^p$: $1/2 < \alpha_k \leq +\infty$ $(k = 1,2,\dots,p)$, where

$p \geqslant 1$, such that all possible intersections $\overline{\Delta}(a_{k_1};\vartheta_{k_1}) \cap \overline{\Delta}(a_{k_2};\vartheta_{k_2})$ $(k_1 \neq k_2)$ of the closed sets $\{\overline{\Delta}(a_k,\vartheta_k)\}_1^p$ contain only the single point—the origin.*

Finally, consider, in the z-plane, the point set

$$M\{\vartheta_1,\dots,\vartheta_p;\alpha_1,\dots,\alpha_p\} \equiv M\{\vartheta;\alpha\} \equiv \bigcup_1^p \Delta(\alpha_k;\vartheta_k), \tag{5}$$

consisting of rays issuing from the origin (if among the numbers α_k there are some equal to $+\infty$), and of angular domains with vertex at the same point $z = 0$ (if among the numbers α_k there are some distinct from $+\infty$).

The complement of the closed set $\overline{M}\{\vartheta;\alpha\}$, evidently, consists of angular domains of the form $\Delta(\rho_k;\psi_k)$, $1/2 < \rho_k < +\infty$ $(k = 1,2,\dots,p)$, $-\pi < \psi_1 < \psi_2 < \dots < \psi_p \leqslant \pi$, and, evidently,

$$\sum_{k=1}^p \left(\frac{1}{\alpha_k} + \frac{1}{\rho_k}\right) = 2. \tag{6}$$

Next denote

$$\rho_M = \max_{1 \leqslant k \leqslant p} \{\rho_k\} \tag{7}$$

and note that this number may serve as a metric characteristic of the complement $C\overline{M}\{\vartheta;\alpha\}$ of the set $\overline{M}\{\vartheta;\alpha\}$, since the opening of the smallest of the angles $\{\Delta(\rho_k;\psi_k)\}_1^p$, complementary to the set $\overline{M}\{\vartheta;\alpha\}$, is equal to π/ρ_M. Here it is evident that $\rho_M > p/2 \geqslant 1/2$.

With the point set $M\{\vartheta;\alpha\}$ of the z-plane we shall associate the closed set

$$\mathcal{E}\{\vartheta;\alpha\} = \bigcup_1^p e\left(|\varphi - \vartheta_k| \leqslant \frac{\pi}{2\alpha_k}\right) \tag{8}$$

—the sum of nonoverlapping intervals $\left[\vartheta_k - \pi/2\alpha_k,\ \vartheta_k + \frac{\pi}{2\alpha_k}\right]$ $(k = 1, 2, \dots, p)$ (degenerating to the point ϑ_k when $\alpha_k = +\infty$). In addition, let us single out the set

$$\mathcal{E}^*\{\vartheta;\alpha\} \subset \mathcal{E}\{\vartheta;\alpha\},$$

consisting of all isolated and interior points of the set $\mathcal{E}\{\vartheta;\alpha\}$. Finally, we shall agree to understand by $\widetilde{\Delta}(\alpha;\vartheta)$ the domain $\Delta(\alpha;\vartheta)$ for $\alpha < +\infty$, and the empty set for $\alpha = +\infty$. Then, if at least one of the numbers $\{\alpha_k\}_1^p$ is distinct from $+\infty$, the open set

$$\widetilde{M}\{\vartheta;\alpha\} = \bigcup_1^p \widetilde{\Delta}(\alpha_k;\vartheta_k) \subset M\{\vartheta;\alpha\} \tag{9}$$

contains, evidently, the totality of all interior points of the set $M\{\vartheta;\alpha\}$.

3°. Let there be in the z-plane an arbitrary set of the form $M\{\vartheta;\alpha\} \equiv M\{\vartheta_1,\dots,\vartheta_p;\alpha_1,\dots,\alpha_p\}$, consisting of a finite number $p \geqslant 1$ of nonoverlapping sets $\{\Delta(\vartheta_k^0;\alpha_k^0)\}_1^p$, and let ρ_M have the preceding meaning (7).

We now denote by $\mathcal{H}^{(\vartheta_1^0,\dots,\vartheta_p^0)}[a_1,\dots,\alpha_p;\omega]$ $(-1 < \omega < 1)$ the class of functions $F(z)$, defined on the set $M\{\vartheta;\alpha\}$ and such that:

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* It is easy to see that this condition is equivalent to the chain of inequalities

$$\vartheta_{k+1} - \vartheta_k > \frac{\pi}{2}\left(\frac{1}{\alpha_k} + \frac{1}{\alpha_{k+1}}\right) \quad (k = 1, 2, \dots, p),$$

where it is put that $\alpha_{p+1} = \alpha_1,\ \vartheta_{p+1} = \vartheta_1 + 2\pi$.

A. For $\varphi \in \mathcal{E}^*\{\vartheta;\alpha\}$

$$I_F(\varphi) = \int_0^{\infty} \left|F\left(re^{i\varphi}\right)\right|^2 r^{\omega}\, dr \leqslant C_F < +\infty,$$

where C_F does not depend on φ.

B. If the set $\widetilde{M}\{\vartheta;\alpha\}$ is nonempty, then the function $F(z)$ is holomorphic in each domain $\Delta(\alpha_k;\vartheta_k)$ $(1/2 < \alpha_k < +\infty)$.

The following main theorem on the parametric representation of the class $\mathcal{H}_2^{(\vartheta_1,\dots,\vartheta_p)}[\alpha_1,\dots,\alpha_p;\omega]$ is established.

Theorem 1. Let the parameters ρ, μ, and γ_k satisfy the conditions

$$\rho \geqslant \rho_M, \qquad \mu = \frac{1+\omega+\rho}{2\rho}, \qquad \frac{1}{\gamma_k} = \frac{1}{\rho} + \frac{1}{\alpha_k}, \quad (k = 1, 2, \dots, p). \tag{10}$$

Then the following assertions are valid:

a) The class $\mathcal{H}_2^{(\vartheta_1,\dots,\vartheta_p)}[\alpha_1,\dots,\alpha_p;\omega]$ coincides with the set of functions admitting a representation of the form

$$\begin{aligned} F\left(r^{1/\rho}e^{i\varphi}\right) = r^{1-\mu} \sum_{k=1}^{p} \Biggl\{ &\frac{d}{dr}\left[r^{\mu} \int_0^{\infty} E_\rho\left(e^{i\pi/2\gamma_k} e^{i\varphi} r^{1/\rho} \tau^{1/\rho};\mu+1\right) v_{(-)}^{(k)}(\tau)\tau^{\mu-1}\, d\tau\right] \\ &+ \frac{d}{dr}\left[r^{\mu} \int_0^{\infty} E_\rho\left(e^{-i\pi/2\gamma_k} e^{i\varphi} r^{1/\rho} \tau^{1/\rho};\mu+1\right) v_{(+)}^{(k)}(\tau)\tau^{\mu-1}\, d\tau\right] \Biggr\}, \end{aligned} \tag{11}$$

$$\varphi \in \mathcal{E}\{\vartheta;\alpha\}, \qquad r \in (0, +\infty),$$

where $\{v_{(\pm)}^{(k)}(t)\}_1^p$ are arbitrary functions from the class $L_2(0,\infty)$.

Here formula (11) defines the function $F\left(r^{1/\rho}e^{i\varphi}\right)$ for every $r \in (0,+\infty)$, if φ is an interior point of the set $\mathcal{E}\{\vartheta;\alpha\}$, and for almost all $r \in (0,+\infty)$, if φ is a boundary or isolated point.

b) If the set $\widetilde{M}\{\vartheta;\alpha\}$ is nonempty, then on it the function defined by formula (11) also admits a representation of the form

$$F(z) = \sum_{k=1}^{p} \left\{ \int_0^{\infty} E_\rho\left(e^{i\pi/2\gamma_k} z\tau^{1/\rho};\mu\right) v_{(-)}^{(k)}(\tau)\tau^{\mu-1}\, d\tau + \int_0^{\infty} E_\rho\left(e^{-i\pi/2\gamma_k} z\tau^{1/\rho};\mu\right) v_{(+)}^{(k)}(\tau)\tau^{\mu-1}\, d\tau \right\}, \quad z \in \widetilde{M}\{\vartheta; \tag{12}$$

4°. The problem of inverting transformation (11), when the function $F(z) \in \mathcal{H}_2^{(\vartheta_1,\dots,\vartheta_p)}[\alpha_1,\dots,\alpha_p;\omega]$ is given and $\{v_{(\pm)}^{(k)}(\tau)\}_1^p$ are the sought functions from the class $L_2(0,+\infty)$, has a unique solution. To formulate the corresponding result, we make several remarks.

Let $F(z) \in \mathcal{H}_2^{(\vartheta_1,\dots,\vartheta_p)}[\alpha_1,\dots,\alpha_p;\omega]$. Then, taking into account the definition of the set $\mathcal{E}^*\{\vartheta;\alpha\}$ and conditions A and B satisfied by the function $F(z)$, we conclude:

If, for a given k $(1 \leqslant k \leqslant p)$, we have $\alpha_k = +\infty$, then the function $F(z)$ is defined and measurable on the ray $\operatorname{Arg} z = \vartheta_k$ and satisfies the condition $I_F(\vartheta_k) \leqslant C_F$.

If, for a given k $(1 \leqslant k \leqslant p)$, we have $\alpha_k < +\infty$, then the function $F(z)$ is holomorphic in the domain $\Delta(\alpha_k;\vartheta_k)$ and satisfies the condition

$$I_F(\vartheta_k+\varphi) = \int_0^\infty \left|F\left(re^{i(\vartheta_k+\varphi)}\right)\right|^2 r^\omega\,dr \leqslant C_F, \qquad |\varphi| < \frac{\pi}{2\alpha_k}.$$

But then, as is known (2), the function $F(z)$ will have boundary values $F\left(re^{i(\vartheta_k\pm\pi/2\alpha_k)}\right)$ almost everywhere on the boundary of the domain $\Delta(\alpha_k;\vartheta_k)$, for which also $I_F(\vartheta_k \pm \pi/2\alpha_k) \leqslant C_F$.

Thus, for the function $F(z)$, or for its boundary values, there exist all the integrals

$$I_F\left(\vartheta_k \pm \frac{\pi}{2\alpha_k}\right) = \int_0^\infty \left|F\left(re^{i(\vartheta_k\pm\pi/2\alpha_k)}\right)\right|^2 r^\omega\,dr \le C_F \quad (k=1,2,\dots,p), \tag{13}$$

where for $\alpha_k = +\infty$ one should set $e^{\pm i\pi/2\alpha_k} = 1$.

Putting, for an arbitrary value $\rho \ge 1/2$, $\mu = (1+\omega+\rho)/2\rho$, we write conditions (13) in the form

$$F\left(e^{i(\vartheta_k\pm\pi/2\alpha_k)}t^{1/\rho}\right)t^{\mu-1} \in L_2(0,+\infty) \quad (k=1,2,\dots,p). \tag{13'}$$

Then with the function $F(z)$ one can associate the collection of functions

$$v_{(\pm)}^{(k)}(\tau;F) = \frac{e^{\pm i\pi/2(1-\mu)}}{2\pi\rho}\frac{d}{d\tau}\int_0^\infty \frac{e^{\pm i\tau t}-1}{\pm it}F\left(e^{i(\vartheta_k\mp\pi/2\alpha_k)}t^{1/\rho}\right)t^{\mu-1}\,dt \tag{14}$$

$(k=1,2,\dots,p)$ of the class $L_2(-\infty,+\infty)$.

Under assumption (10) the following is established.

Theorem 2. If $F(z) \in \mathcal{H}_2^{\rho(\vartheta_1,\dots,\vartheta_p)}[\alpha_1,\dots,\alpha_p;\omega]$, then in the representation (11) of Theorem 1 the functions $\{v_{(\pm)}^{(k)}(\tau)\}_1^p$ are unique and almost everywhere are determined by the formulas

$$v_{(+)}^{(k)}(\tau) = v_{(+)}^{(k)}(\tau; F), \qquad v_{(-)}^{(k)}(\tau) = v_{(-)}^{(k)}(\tau; F) \quad (k = 1, 2, \dots, p). \tag{15}$$

5°. Suppose again that $F(z) \in \mathcal{H}_2^{\rho(\vartheta_1,\dots,\vartheta_p)}[\alpha_1, \dots, \alpha_p; \omega]$ and that the functions $v_{(\mp)}^{(k)}(\tau; F)$ are defined by formulas (14). We introduce the functions

$$L_k^{(\pm)}(z; F; \sigma) = \int_0^\infty E_\rho\left(e^{\mp i\pi/2\gamma_k\sigma} z\tau^{1/\rho}; \mu\right) v_{(\mp)}^{(k)}(\tau; F)\tau^{\mu-1}\, d\tau \quad (k = 1, 2, \dots, p), \tag{16}$$

which are entire functions of growth (ρ, σ).* From Theorems 1 and 2, under the same conditions (10), the following approximation theorem follows.

Theorem 3. Let $F(z) \in \mathcal{H}_2^{\rho(\vartheta_1,\dots,\vartheta_p)}[\alpha_1, \dots, \alpha_p; \omega]$, and let $\mathcal{E}\{\vartheta; \alpha\}$ be a closed collection associated with the set of its definition $M\{\vartheta; \alpha\}$.

Then for every $\rho \geq \rho_M$ the entire functions of growth (ρ, σ)

$$F_\rho(z; \sigma) = \sum_{k=1}^{p} \left\{ L_k^{(+)}\left(e^{-i\vartheta_k} z; F; \sigma\right) + L_k^{(-)}\left(e^{-i\vartheta_k} z; F; \sigma\right) \right\} \tag{17}$$

approximate in the mean the function $F(z)$ on the closed set $\overline{M}\{\vartheta; \alpha\}$ in the sense that

$$\lim_{\sigma \to +\infty} \left\{ \sup_{\varphi \in \mathcal{E}\{\vartheta; \alpha\}} \int_0^\infty \left| F(re^{i\varphi}) - F_\rho(re^{i\varphi}; \sigma) \right|^2 r^\omega \, dr \right\} = 0. \tag{18}$$

In conclusion we note that various special cases of the general theorems formulated above, under particular assumptions concerning the set $M\{\vartheta; \alpha\}$ and the definition of the class $\mathcal{H}_2^{\rho(\vartheta_1,\dots,\vartheta_p)}[\alpha_1, \dots, \alpha_p; \omega]$, are of independent interest; however, we shall not enumerate them.

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CITED LITERATURE

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3. R. Paley, R. Wiener, *Fourier Transforms in the Complex Domain*, N. Y., 1934.

* In other words, their order is $\leq \rho$, and in the case of equality the type is $\leq \sigma$.

Note: Figure translations are in progress. See original paper for figures.

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