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Abstract

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ON SOME PROPERTIES OF ANALYTIC FUNCTIONS OF THE CLASS $H_p^{'}$

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By the class H_p' we shall mean the set of analytic functions $f(z)$ in the disk $|z| < 1$ such that

$$H_p'(f) = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r\, dr\, d\theta < \infty, \qquad p > 0.$$

This class is a natural generalization of the class H_p of F. Riesz, which is defined as the set of functions analytic in the disk such that

$$H_p(f) = \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta < \infty.$$

The question naturally arises of the relation between the classes H_p' and H_p.

1°. **Theorem 1.** If $f'(z) \in H_1'$, then $f(z) \in H_1$.

An example constructed by S. N. Mergelyan ([1]) shows that there exists a function $f_0(z) \in H_1$ for which $f_0'(z) \notin H_1'$.

To a certain extent the following assertion is the converse of Theorem 1.

Theorem 2. If $f(z) \in \bigcap_{p<1} H_p$, then $f'(z) \in \bigcap_{p<1} H_p'$.

Proof. Since $f(z)$ is analytic in the disk $|z| < 1$, we have

$$f'(z) = \frac{1}{2\pi i} \int_{|t|=R} \frac{f(t)}{(t-z)^2}\, dt.$$

Hence, for $0 < r < R < 1$, one obtains the inequality

$$\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\varphi})| \int_0^{2\pi} \frac{d\theta}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, d\varphi,$$

from which it follows that

$$\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \le \frac{1}{R-r}\int_0^{2\pi} |f(Re^{i\varphi})|\, d\varphi. \qquad (1)$$

Using the known estimate ([2]), for $0 < \delta < 1$,

$$\int_0^{2\pi} |f(Re^{i\theta})| \le C(1-R)^{1-1/\delta}\{H_\delta(f)\}^{1/\delta},$$

where the constant C does not depend on R, we obtain from (1)

$$\int_0^{2\pi} |f'(Re^{i\theta})|\, d\theta \le \frac{C_1}{(1-R)^{1/\delta-1}(R-r)}\{H_\delta(f)\}^{1/\delta}.$$

Applying to the left-hand side of (2) Hölder' s inequality for $p < 1$ ([3]) and putting $r = R^2$, we shall have

$$\left\{\int_0^{2\pi} |f'(R^2e^{i\theta})|^p\, d\theta\right\}^{1/p} \le \frac{C_3}{R(1-R)^{1/\delta}}\{H_\delta(f)\}^{1/\delta}.$$

Raising both sides to the power p and integrating with respect to R from zero to one, we obtain

$$\int_0^1\int_0^{2\pi} |f'(R^2e^{i\theta})|^p R^2\, dR^2\, d\theta \le A(p,f)\int_0^1 \frac{R^2\, dR^2}{R^p(1-R)^{p/\delta}}.$$

Thus, $f'(z) \in H'_p$ for every $p < \delta$ and, consequently, $f'(z) \in \bigcap_{p<1} H'_p$. The theorem converse to Theorem 2 does not hold. This follows from the following assertion.

Theorem 3. *For every $0 < \delta < 1$ there exists a function $f_0(z) \in H_\delta$ for which $f_0(z) \notin H_{\delta+\varepsilon}$ for every $\varepsilon > 0$, but $f'_0(z) \in \bigcap_{p<1} H'_p$.*

The proof is based on a theorem obtained by V. P. Khavin in ([4]). It is interesting to note that V. P. Khavin' s theorem does not extend to the class H'_p. Here the following theorem holds:

Theorem 4. *Let $f(z) \in H'_p$, $p > 0$, but $f(z) \notin H'_{p+\varepsilon}$ for every $\varepsilon > 0$; if $\max|f(z)| = O(\varphi(r))$, where $\varphi(r)$ is a continuous nondecreasing function, $z = re^{i\theta}$, then $\int_0^1 \varphi^q(r)\, dr = \infty$ for all $q > p$.*

For functions belonging to H_p, $p < 1$, the following is true.

Theorem 5. *If $f(r) \in H_p$, $p < 1$, then $f'(z) \in H'_q$, where $q < 2p/(p+1)$.*

We first prove the theorem for functions having no zeros in the disk $|z| < 1$. Let $F(z) \neq 0$ in the disk $|z| < 1$; then for $\varepsilon > 0$ and $|z| < R < 1$

$$[F(z)]^\varepsilon = \frac{1}{2\pi i} \int_{|t|=R} \frac{[F(t)]^\varepsilon}{(t-z)}\, dt,$$

therefore,

$$\varepsilon [F(z)]^{\varepsilon-1} F'(z) = \frac{1}{2\pi i} \int_{|t|=R} \frac{[F(t)]^\varepsilon}{(t-z)^2}\, dt.$$

Hence, for $z = re^{i\varphi}$, $t = Re^{i\theta}$, we have the estimate

$$|F'(z)| \leq \frac{1}{2\pi\varepsilon} |F(z)|^{1-\varepsilon} \int_0^{2\pi} \frac{|F(Re^{i\theta})|^\varepsilon R}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, d\theta. \tag{3}$$

Using the known inequality ([2]) for $F(z) \in H_p$,

$$|F(z)| \leq \frac{C}{(1-r)^{1/p}}, \qquad |z| = r, \tag{4}$$

we derive from (3)

$$|F'(z)| \leq \frac{C_1 R\, |F(z)|^{1-\varepsilon}}{(1-R)^{\varepsilon/p}(R^2 - r^2)}.$$

Raise both sides of the inequality to the power q, $p < q < 1$, and suppose that $r = R^2$; then

$$\int_0^{2\pi} |F'(R^2 e^{i\varphi})|^q\, d\varphi \leq \frac{C_1^q}{R^q (1-R)^{(1+\varepsilon/p)q}} \int_0^{2\pi} |F(Re^{i\theta})|^{(1-\varepsilon)q}\, d\theta. \tag{5}$$

Using (4) we easily obtain an estimate, valid for the whole class H_p and for any $p > 0$, $\alpha > p$:

$$\int_0^{2\pi} \left|F\left(re^{i\varphi}\right)\right|^\alpha d\varphi \leq \frac{C_2}{(1-r)^{(\alpha-p)/p}} H_p(F). \tag{6}$$

We shall now assume that $\varepsilon < 1 - p/q$; applying inequality (6) to the right-hand side of (5), we obtain

$$\int_0^{2\pi} \left|F'\left(R^2 e^{i\theta}\right)\right|^p d\theta \leq \frac{C_3}{(1-R)^{q(1+1/p)-1} R^q}.$$

If the condition $q < \dfrac{2}{1/p+1}$ is satisfied, then $F'(z) \in H'_q$.

If $f(z)$ is an arbitrary function in $H_p,\ p<1$, then it is representable in the form $f(z) = F_1(z) - F_2(z)$ ((5), p. 540), where $F_j(z)$ $(j=1,2)$ belong to H_p and have no zeros in the disk $|z|<1$; this case reduces to the preceding one.

The number $2p/(p+1)$ cannot be replaced by a larger number, as is shown by the example of a function of the form $z^2(1-z)^{-1/p}\left(\ln \dfrac{1}{1-z}\right)^2$.

2°. The question of the zeros of functions of the class H'_p was considered by M. M. Dzhrbashyan ([6]). The following result is due to him.

Let $f(z) \in A(\alpha)$, i.e., let $f(z)$ be analytic in the disk $|z|<1$, $f(z) \not\equiv 0$, $\alpha > -1$, and

$$\frac{\alpha+1}{\pi} \int_0^1 \int_0^{2\pi} \left(1-\rho^2\right)^{\alpha} \ln^+ \left|f\left(\rho e^{i\theta}\right)\right| \rho \, d\rho \, d\theta < \infty;$$

if $a_1, a_2, \ldots$ are the zeros of $f(z)$, counted with multiplicity, and

$$a_1 \le a_2 \le a_3 \le \cdots,$$

then

$$\sum_{k=1}^{\infty} \left(1-|a_k|\right)^{2+\alpha} < \infty.$$

It turns out that if a function belongs to the class $H'_p,\ p>0$, then the following is true.

Theorem 6. *If $f(z) \in H'_p,\ p>0$, and $a_1, a_2, \ldots$ are the zeros of $f(z)$, counted with multiplicity, with*

$$|a_1| \le |a_2| \le |a_3| \le \cdots,$$

then for every $\varepsilon > 0$

$$\sum_{k=1}^{\infty} \left(1-|a_k|\right)^{1+\varepsilon} < \infty.$$

With the aid of one result ([7]) it follows that there exists a function $f(z) \in H'_p,\ 0<p<1$, such that

$$\sum_{k=1}^{\infty}(1-|a_k|)=\infty,$$

where $a_1, a_2, ...$ are the zeros of the function $f(z)$.

3°. The following theorem of G. M. Goluzin is known ([8]).

If

$$f(z)=\sum_{k=0}^{\infty}a_k z^k$$

is an analytic function in the disk $|z|<1$ and $H_1(f)\leq 2\pi$, then for all $n=0,1,2,...$ the estimate $|a_n|\leq 1$ holds.

In this case the equality sign for a certain n is attained if and only if

$$f(z)=\varepsilon\sum_{k=0}^{n}a_k z^k\sum_{k=0}^{n}\bar{a}_k z^{n-k},$$

where for a_k and ε the condition

$$|\varepsilon|\sum_{k=0}^{\infty}|a_k|^2$$

is satisfied.

With respect to the coefficients of functions belonging to the class H_1', one may assert:

Theorem 7. If $f(z)=\sum_{k=0}^{\infty}a_k z^k$ is an analytic function in the disk $|z|<1$ and $H_1'(f)=2\pi$, then for all $n=0,1,2,...$ the estimate $|a_n|\leq n+2$ is valid. In this case the equality sign for a certain n is attained if and only if

$$f(z)=e^{i\alpha}(n+2)z^n,$$

where α is an arbitrary real number.

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Note: Figure translations are in progress. See original paper for figures.