

GROUP CLASSIFICATION OF EQUATIONS DESCRIBING ONE-DIMENSIONAL UNSTEADY GAS FLOWS

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Abstract

Full Text

HYDROMECHANICS

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**GROUP CLASSIFICATION OF EQUATIONS
DESCRIBING ONE-DIMENSIONAL UN-
STEADY GAS FLOWS**

(Presented by Academician L. I. Sedov on 15 I 1964)

The idea of the systematic application of group theory in gas dynamics belongs to L. V. Ovsyannikov and his students (¹⁻⁵). He also gave the group classification of one-dimensional flows with plane waves. In the present work we shall consider one-dimensional flows with an arbitrary dependence of the cross-sectional area of the flow tube on the coordinate. We shall assume the flow to be adiabatic.

Consider the following system of equations describing one-dimensional unsteady flows in the hydraulic approximation:

$$u_t + uu_x + \frac{p_x}{\rho} = 0; \tag{1}$$

$$p_t + up_x + Au_x + uAF_x = 0; \tag{2}$$

$$\rho_t + u\rho_x + \rho u_x + u\rho F_x = 0. \tag{3}$$

Here $F(x) = \ln f(x)$; $A = \rho S_\rho / S_p$; $f(x)$ is the cross-sectional area of the flow tube; p is pressure; ρ is density; u is velocity; S is entropy; for a perfect gas $A = \gamma p$, γ is the adiabatic exponent.

From the system (1)–(3), for $f(x) = x^k$ (where $k = 0, 1, 2$) we obtain equations describing one-dimensional flows with plane waves ($k = 0$), as well as flows with cylindrical ($k = 1$) and spherical ($k = 2$) symmetry.

We note that the equivalence transformations for the system (1)–(3) have the form:

$$p_1 = \lambda p + p_0; \quad \rho_1 = \lambda \rho; \quad A_1 = \lambda A; \quad F_1 = F(a_0 x + b_0) + F_0, \tag{3'}$$

where $\lambda, a_0, b_0, p_0, \dots$ are arbitrary constants.

We shall regard $F(x)$ and $A(p, \rho)$ as two arbitrary functions, whose arbitrariness may affect the dimension of the group admitted by the system (1)–(3). To

determine the group of transformations admitted by the system (1)–(3), we write, as usual, the admitted operator in the form:

$$X = \xi^0 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \sigma \frac{\partial}{\partial p} + \tau \frac{\partial}{\partial \rho}. \quad (4)$$

To obtain equations for the coefficients of the operator (4), we construct the system of determining equations (6). This system has the following form:

$$\xi_p^i = \xi_\rho^i = \xi_u^i = \xi_x^0 = \eta_p = \eta_\rho = \tau_u = \tau_p = \sigma_u = \sigma_\rho = 0; \quad (5a)$$

$$\eta = u\xi_x^1 - u\xi_t^0 + \xi_t^1; \quad \tau = \rho(\sigma_p + 2\xi_t^0 - 2\xi_x^1); \quad (5b)$$

$$A\sigma_\rho = \sigma A_p + \tau A_\rho; \quad (6)$$

$$\eta_t + u\eta_x + \sigma_x/\rho = 0; \quad (7)$$

$$\sigma_t + u\sigma_x + A\eta_x = -W_1; \quad (8)$$

$$\tau_t + u\tau_x + \rho\eta_x = -W_2; \quad (9)$$

here

$$W_1 = A\widetilde{W}; \quad W_2 = \rho\widetilde{W}; \quad \widetilde{W} = u\xi_{tF}^0 x + \eta F_x + u\xi^1 F_{xx}. \quad (10)$$

From the system (5)–(9) we find

$$\xi^0 = bt^2 + (a + c_1)t + e, \quad \xi^1 = bxt + ax + dt + c, \quad (11)$$

$$\sigma = pm(t) + n(t); \quad (11)$$

$$m_t + 3b = -\widetilde{W}; \quad (12)$$

$$n_t + b(A - 3p) = \widetilde{W}(p - A), \quad (13)$$

where $m(t)$, $n(t)$ are new unknown functions; a, b, e, d, c_1 are constants.

Let us examine more closely the expression for \widetilde{W} :

$$\widetilde{W} = \xi_t^1 F_x + u(\xi^1 F_{xx} + \xi_x^1 F_x) = \widetilde{W}_1 + u\widetilde{W}_2. \quad (14)$$

Since $m(t)$ depends only on t , and does not depend on u , from comparison of (12) and (14), using the expression for ξ^1 , we obtain

$$\widetilde{W}_2 = t[b(xF_{xx} + F_x) + dF_{xx}] + [a(xF_{xx} + F_x) + cF_{xx}] = tQ_0 + Q_1 = 0,$$

where the notation is obvious. The condition $\widetilde{W}_2 = 0$ gives $Q_0 = Q_1 = 0$.

In finding $F(x)$ up to equivalence, it is sufficient to consider the following cases:

1. $b, d(a, c)$ arbitrary. Then $x F_{xx} + F_x = 0$; $F_{xx} = 0$. Hence $F = \text{const}$, which (see (3')) may be taken equal to zero, corresponding to $f(x) = 1$. This case was studied in detail in paper ⁽⁶⁾.
2. $b(a)$ arbitrary, $d(c) = 0$. Then $x F_{xx} + F_x = 0$. Hence $F(x) = k \ln k_1 x$. Using (3'), this can be written in the form $F(x) = \ln x^k$, which corresponds to

$$f(x) = x^k. \quad (15)$$

Both cases are combined by one formula $f(x) = x^k$, where k is any number or zero.

3. $b(a) = 0$, $d(c)$ arbitrary. Then $F_{xx} = 0$. Hence $F = a_1 x + b_1$. Using (3'), we write: $F(x) = x$. This corresponds to

$$f(x) = e^x. \quad (16)$$

4. $b = d = a = c = 0$. In this case $F(x)$ may be arbitrary.

Let us consider the case $f(x) = x^k$. For $f(x) = x^k$, the condition $\widetilde{W}_2 = 0$ is satisfied identically, and $\widetilde{W}_1 = kb$. The system (12)–(13) takes the form

$$m_t = -b(3 + k); \quad (17)$$

$$n_t = b[(3 + k)p - A(1 + k)]. \quad (18)$$

The construction of the operator both for $b = 0$ and for $b \neq 0$ is carried out simply. An extension of the group for $b \neq 0$, as is seen from (18), occurs (we assume that $k \neq -1$) when

$$A = p \frac{3 + k}{1 + k}, \quad \text{i.e. when } \gamma = \frac{3 + k}{1 + k}. \quad (18a)$$

This formula is a generalization of the analogous formula obtained in ⁽⁶⁾ for integer values of k .

Omitting the calculations, we write the operator X :

$$X = [bt^2 + (a + c_1)t + e] \frac{\partial}{\partial t} + (bxt + ax) \frac{\partial}{\partial x} + (bx - c_1 u - but) \frac{\partial}{\partial u} - [b(3 + k)t - m_0] p \frac{\partial}{\partial p} - [b(1 + k)t - m_0 - 2c_1] \rho \frac{\partial}{\partial \rho}.$$

Here a, b, c_1, m_0, e are arbitrary constants.

For $f(x) = e^x$ the analysis is carried out analogously. Let us only note that the system (12)–(13) takes the form:

$$m_t = -d; \quad n_t = d(p - A),$$

and, consequently, for $A = p$ (i.e., $\gamma = 1$) the constant d is preserved. In the case of arbitrary $f(x)$, only the constants c_1, m_0, e remain in the operator. Thus, for $f(x) = x^k$ the system (1)–(3) admits the widest group.

Let us now note that the values $k = 0, -1$ are special. Indeed, substituting into (8) the value \widetilde{W} from (10) and η_x from (5b), we reduce (8) to the form

$$\sigma_t + u [\sigma_x + A (\xi^1 F_{xx} + \xi_x^1 F_x) + A (\xi_{xx}^1 - \xi_{tx}^1)] + A (\xi_{tx}^1 + \xi_t^1 F_x) = 0. \quad (19)$$

Taking into account the conditions $\xi_{xx}^1 = \xi_{tt}^1 = 0$, which are easily obtained from (7), and also the fact that $\sigma_u = 0$, we immediately obtain from (19)

$$\sigma_x + A (\xi^1 F_{xx} + \xi_x^1 F_x) = 0.$$

Hence it follows that

$$A_\rho (\xi^1 F_{xx} + \xi_x^1 F_x) = 0. \quad (20)$$

After this, from equation (19), using (5a), we obtain one more condition

$$A_\rho (\xi_x^1 + \xi^1 F_x) = 0. \quad (21)$$

The system (20), (21) reduces to the single equation

$$\xi^1 A_\rho (F_{xx} - F_x^2) = 0. \quad (22)$$

This equation is interesting in that it relates the functions $A(p, \rho)$ and $F(x)$, which, it would seem, are in no way related to each other. For $F(x) = k \ln x$, condition (22) has the form

$$\xi^1 A_\rho (k + 1)k = 0. \quad (23)$$

From (23) it is obvious that if $k \neq 0, -1$, then either $\xi^1 = 0$, or $A_\rho = 0$. For $k = 0, -1$ condition (23) imposes no restrictions on A_ρ .

Let $k \neq 0, -1$. Then $A_\rho = 0$. In the system (17)–(18) either $b = 0$ and $A = A(p)$, or $b \neq 0$ and $A = p^{\frac{3+k}{1+k}}$. For $b = 0$, as is easy to see, $\sigma = m_0 p + n_0$ and (6)

takes the form $m_0(pA_p - A) + n_0A_p = 0$. Hence we obtain: a) if $m_0 = n_0 = 0$, then $A = A(p)$ is arbitrary; b) if $m_0 = 0$, $n_0 \neq 0$, then $A = 1$ (see (3')); c) if $m_0 \neq 0$, $n_0 = 0$, then $A = \gamma p$.

When applying group theory to the solution of equations, what is of interest is not the form of the operator X itself, but the so-called system of optimal subgroups [6]. Omitting all the calculations, we obtain the following system of optimal subgroups H_i of rank 1 for nonsingular values of γ from (18a):

$$H_1 = X_1 + \varepsilon X_4; \quad H_2 = X_2 + \beta X_4; \quad H_3 = X_3 + \beta X_4;$$

$$H_4 = X_2 + \alpha X_3 + \beta X_4; \quad H_5 = X_1 + X_2 - X_3 + \beta X_4, \quad H_6 = X_4;$$

where

$$X_1 = \partial/\partial t, \quad X_2 = t\partial/\partial t + x\partial/\partial x; \quad X_3 = t\partial/\partial t + 2\rho\partial/\partial\rho - u\partial/\partial u;$$

$$X_4 = p\partial/\partial p + \rho\partial/\partial\rho; \quad \varepsilon = 0, 1; \quad \alpha, \beta \text{ arbitrary.}$$

The system of invariants for H_i , as well as the corresponding system of ordinary differential equations, is found easily.

For example, for the operator $H_4(\alpha = -1) = X_2 - X_3 + \beta X_4$, where the solution can be sought in the form $u = xV(t)$, $\rho = x^{\beta-2}R(t)$, $p = x^\beta P(t)$ (V, R, P are new unknown functions), system (1)–(3) reduces to a single equation

$$\frac{dV}{V} = -\frac{(1 + \beta z)}{(2\beta z + 2 - B)} \frac{dz}{z}.$$

Here $B = \gamma(1 + k) + 1 - k$; $z = P/RV$. This equation is solved elementarily, and the dependence $z(t)$, as is not difficult to show, is found from

$$t = \int z^{(1-B)/(2-B)} (2\beta z + 2 - B)^{(4-B)/2(2-B)} dz.$$

For $\gamma = (3 + k)/(1 + k)$ the integral is taken in quadratures, after which it is not difficult to find expressions for u, p, ρ .

In conclusion, I consider it my pleasant duty to express gratitude to L. V. Ovsyannikov for critical remarks and a very valuable discussion of the work.

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