MATHEMATICS

S. A. TERSENOV

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. A. TERSENOV

THE RIEMANN-HILBERT PROBLEM FOR A SYSTEM OF EQUATIONS DEGENERATING ON THE BOUNDARY

(Presented by Academician M. A. Lavrent' ev, November 23, 1963)

As is known (see ([1])), the problem of finding a solution of the system of equations of elliptic type

$$\begin{aligned} u_x - v_y + au + bv = f_1, \\ yu_y - v_x + cu + dv = f_2 \end{aligned} \qquad (y > 0) \tag{1}$$

in a domain D containing a segment AB of the x-axis (the line of degeneration), satisfying on the boundary Γ a condition of the form $\lambda_1 u + \lambda_2 v = f$, is, generally speaking, ill-posed. In the present article this problem is investigated in the modified formulation proposed in ([1]).

Let the coefficients and the right-hand sides of system (1) be Hölder-continuous functions in $D + \Gamma$, and let

$$c(x, y) = \alpha + yc_0(x, y), \tag{2}$$

where α is a constant; the functions λ_i, f are continuous, with λ_1 and λ_2 not vanishing simultaneously, and $\lambda_1 \neq 0$ on AB, so that without loss of generality one may assume $\lambda_1^2 + y\lambda_2^2 = 1$ on Γ. By means of a nonsingular change of independent variables one can arrange that $\Gamma = \sigma + AB$, where σ is the parabola $x^2 + 4y = 1$, and AB is the segment $(-1, 1)$ of the x-axis.

We rewrite system (1) in complex form:

$$S_\alpha(w) + Aw + B\overline{w} = F, \tag{3}$$

where

$$w = u + iv, \qquad F = f_1 + if_2; \qquad 2A = a + d - ib + iyc_0;$$

$$2B = a - d + ib - iyc_0.$$

In what follows, equation (3) with $A = B = F \equiv 0$ will be denoted by (3_0).

Let $2\pi n$ be the increment of the argument of the function $\lambda_1 + i\sqrt{y}\,\lambda_2$ when the contour Γ is traversed once in the positive direction. Then, by multiplying the function $\lambda_2 + i\lambda_1$ by a completely determined function $\nu(x, y)$, nowhere equal to zero on Γ, one can arrange that the function $\nu(\lambda_2 + i\lambda_1)$ be the boundary value from D, on Γ, of some function $\varphi(x, y)$ satisfying equation (3_0) for $2\alpha = 1$, where at a fixed point $(0, y_0) \in D$ the function $\varphi = O(r^{-n})$, $r^2 = x^2 + 4(\sqrt{y} - \sqrt{y_0})^2$. Therefore, without restricting generality, we shall assume that the functions λ_i are defined in the whole domain $D + \Gamma$ and that $\lambda_2 + i\lambda_1 = \varphi$. By a solution of equation (3) we shall mean a function w, continuously differentiable in D, satisfying (3).

Theorem 1. Let $\alpha < 0$.

1) If $n \geq 0$, then there always exists a solution, continuous in $D + \Gamma$, of equation (3) satisfying the condition

$$\operatorname{Im}(\varphi w) = f \quad \text{on } \Gamma, \tag{4}$$

and the homogeneous problem ($F = f = 0$) has $2n + 1$ linearly independent solutions.

2) For $n < 0$ the homogeneous problem has only the trivial solution, and for the existence of a solution of the nonhomogeneous problem it is necessary and sufficient that $-2n - 1$ conditions of the form

$$\operatorname{Re}\left\{\iint_D \chi_j' F \, d\xi dt\right\} + \int_\Gamma^\circ f\chi_j'' \, ds = 0, \qquad j = 1, \dots, -2n-1, \tag{5}$$

be fulfilled, where χ_j', χ_j'' are linearly independent functions.

Theorem 2. Let $\alpha < 1$.

1) If $n \geq 0$, then there always exists a solution of equation (3), continuous in $D + \Gamma$, satisfying the condition

$$\operatorname{Re}\{w[(y+1)\varphi + (y-1)\overline{\varphi}]\} = f \quad \text{on } \Gamma.$$

The homogeneous problem has $2n + 1$ linearly independent solutions for $\alpha \leq 0$ and $2n$ linearly independent solutions for $0 < \alpha < 1$.

2) If $n < 0$, then the homogeneous problem has only the trivial solution, and for the existence of a solution of the nonhomogeneous problem it is

necessary and sufficient that $-2n-1$ conditions of the form (5) be fulfilled if $\alpha \leq 0$, and $-2n$ conditions if $0 < \alpha < 1$.

Theorem 3. Let $\alpha \geq 1$.

1) For $n \geq 0$ there always exists a solution of equation (3), continuous in $D + \sigma$ and bounded in $D + \Gamma$, satisfying one of the conditions:

$$\operatorname{Im}(\varphi w) = f \quad \text{or} \quad \operatorname{Re}\{w[(y+1)\varphi + (y-1)\overline{\varphi}]\} = f \quad \text{on } \sigma.$$

The homogeneous problem corresponding to the first boundary condition has $2n + 1$ linearly independent solutions, while the homogeneous problem corresponding to the second boundary condition has $2n + 1$ linearly independent solutions for $\alpha > 1$ and $2n$ linearly independent solutions for $\alpha = 1$.

2) For $n < 0$ the homogeneous problems have only the trivial solution, and for the existence of solutions of the nonhomogeneous problems it is necessary and sufficient that $-2n - 1$ conditions of the form (5) be fulfilled in the case of the first boundary condition, and $-2n$ conditions for $\alpha = 1$ and $-2n - 1$ for $\alpha > 1$ in the case of the second boundary condition.

This theorem, for $\alpha = 1,\ a = b = d = c_0 \equiv 0$, in the case of the second boundary condition, is contained in the work [(2)].

Theorem 4. Let $\alpha \geq 1$.

1) If $n \geq 0$, then there always exists a solution of equation (3), continuous in $D + \sigma$, satisfying the condition

$$y^\alpha \operatorname{Im}(\varphi w) = f \quad \text{on } \Gamma,$$

where the functions $y^\alpha \operatorname{Re} w$ and $y^{\alpha-1} \operatorname{Im} w$ are continuous in $D + \Gamma$. The homogeneous problem has $2n + 1$ linearly independent solutions.

2) If, however, $n < 0$, then the solution is always unique, and for its existence it is necessary and sufficient that $-2n - 1$ conditions of the form (5) be fulfilled.

Theorem 5. Let $\alpha > 1$.

1) For $n \geq 0$ there always exists a solution of equation (3), continuous in $D + \sigma$, satisfying the condition

$$y^{\alpha-1} \operatorname{Re}\{w[(y+1)\varphi + (y-1)\overline{\varphi}]\} = f \quad \text{on } \Gamma,$$

where the functions $y^\alpha \operatorname{Re} w$ and $y^{\alpha-1} \operatorname{Im} w$ are continuous in $D + \Gamma$. The homogeneous problem has $2n + 1$ linearly independent solutions.

2) If, however, $n < 0$, then the solution is always unique, and for its existence it is necessary and sufficient that $-2n-1$ conditions of the form (5) be fulfilled.

We shall give the proof of Theorem 1. The remaining theorems are proved in approximately the same way.

Introduce the notation:

$$E_\nu = \frac{\Gamma(\nu-\frac{1}{2})\Gamma(\nu)}{2^{2-2\nu}\sqrt{\pi}\Gamma(2\nu-1)}(ty)^{(1-\nu)/2}\int_0^\infty e^{-|x-\xi|z}J_{\nu-1}(2\sqrt{y}\,z)J_{\nu-1}(2\sqrt{t}\,z)\,dz,$$

$$M_\nu = \frac{\Gamma(\nu-\frac{1}{2})\Gamma(\nu)}{2^{2-2\nu}\sqrt{\pi}\Gamma(2\nu-1)}t^{(1-\nu)/2}y^\nu\,\mathrm{sign}(\xi-x)\cdot\int_0^\infty e^{-|x-\xi|z}J_\nu(2\sqrt{y}\,z)J_{\nu-1}(2\sqrt{t}\,z)\,dz,$$

$$H = y-\alpha\left[E_{1-\alpha}(x,y;\xi,t)-\rho^{2\alpha-1}E_{1-\alpha}\left(x,y;\frac{\xi}{\rho^2},\frac{t}{\rho^4}\right)\right]+$$
$$+\,i\left[M_{1-\alpha}(x,y;\xi,t)-\rho^{2\alpha-1}M_{1-\alpha}\left(x,y;\frac{\xi}{\rho^2},\frac{t}{\rho^4}\right)\right],$$

$$2\Phi_1 = -y^{-\alpha}E_{1-\alpha,x}+y^{1-\alpha}E_{2-\alpha,x}+iy^{1-\alpha}E_{1-\alpha,y}-i\left(y^{1-\alpha}E_{2-\alpha}\right)_y,$$

$$2\Phi_2 = -y^{-\alpha}E_{1-\alpha,x}-y^{1-\alpha}E_{2-\alpha,x}+iy^{1-\alpha}E_{1-\alpha,y}+i\left(y^{1-\alpha}E_{2-\alpha}\right)_y,$$

$$\Psi_1 = \Phi_1-\frac{1}{2}\int_\delta\left[\Phi_1(\xi_1,t_1;\xi,t)+\overline{\Phi_2(\xi_1,t_1;\xi,t)}\right](t_1H_{t_1}\,d\xi_1-H_{\xi_1}\,dt_1)+$$
$$+\frac{\alpha}{2}\int_{-1}^1\left[\Phi_1(\xi_1,0;\xi,t)+\overline{\Phi_2(\xi_1,0;\xi,t)}\right]H(x,y;\xi_1,0)\,d\xi_1,$$

$$\Psi_2 = \Phi_2-\frac{1}{2}\int_\sigma\left[\Phi_2(\xi_1,t_1;\xi,t)+\overline{\Phi_1(\xi_1,t_1;\xi,t)}\right](t_1H_{t_1}\,d\xi_1-H_{\xi_1}\,dt_1)+$$
$$+\frac{\alpha}{2}\int_{-1}^1\left[\Phi_2(\xi_1,0;\xi,t)+\overline{\Phi_1(\xi_1,0;\xi,t)}\right]H(x,y;\xi_1,0)\,d\xi_1,$$

$$K(F) = \iint_D\left[\Psi_1(x,y;\xi,t)F(\xi,t)+\Psi_2(x,y;\xi,t)\overline{F(\xi,t)}\right]d\xi\,dt,$$

$$\Omega(\mu) = \int_\sigma\mu\,(tH_t\,d\xi-H_\xi\,dt)-\alpha\int_{-1}^1\mu H(x,y;\xi,0)\,d\xi,$$

$$T(\Psi) = i\,\frac{[(y-1)\varphi+(y+3)\overline{\varphi}]\Psi+(y-1)(\varphi+\overline{\varphi})\overline{\Psi}}{(y-1)(\varphi^2+\overline{\varphi}^2)+2(1+y)|\varphi|^2},$$
$$\rho^2 = \xi^2+4t.$$

For the properties of the functions E_ν and M_ν used here, see [3].

The function $\Omega(\mu)$ is a solution of equation (3_0) satisfying on Γ the condition $\operatorname{Re}\Omega = \mu$. It is easy to verify that, if $T(\Psi)$ is a solution of equation (3_0), then the function Ψ will be a solution of the equation

$$\mathcal{L}_\alpha(\Psi) + A_0\Psi + B_0\overline{\Psi} = 0, \tag{6}$$

where A_0 and B_0 are expressed in terms of φ and α. Conversely, if Ψ is a solution of equation (6), then $T(\Psi)$ will be a solution of (3_0), with $\operatorname{Im}(\varphi T(\Psi)) = \operatorname{Re}\Psi$ on Γ.

For any continuous function μ, the equation

$$\Psi + K(A_0\Psi + B_0\overline{\Psi}) = \Omega(\mu) \tag{7}$$

always has, and moreover has uniquely, a solution satisfying the condition $\operatorname{Re}\Psi = \mu$ on Γ.

Let $n \geqslant 0$. If w is a solution of equation (3) satisfying condition (4), then it satisfies the integral equation

$$w + K(Aw + B\overline{w}) - T(\Psi) = K(F) + T(\Psi_0) + \Phi, \tag{8}$$

where $\Psi(w)$ and Ψ_0 are solutions of the integral equation (7), i.e., also of equation (6), for μ equal to $\operatorname{Im}\{\varphi K(Aw + B\overline{w} - F)\}$ and f, respectively, and Φ is the general solution of equation (3_0) satisfying the homogeneous

condition corresponding to (4). The integral equation (8) is always solvable. It follows from equation (8) that the number of linearly independent solutions of the homogeneous problem is equal to the number of linearly independent solutions of equation (3_0) under the homogeneous conditions corresponding to (4). We shall prove that this number is equal to $2n+1$. For this purpose consider the functions

$$V_k = \alpha_k \sum_{\nu=0}^{k} C_k^\nu x^\nu i^{k-\nu} p_{k-\nu} + \beta_k \sum_{\nu=0}^{k} C_k^\nu x^\nu i^{k-\nu} q_{k-\nu}, \qquad k = 0, 1, \dots, n-1,$$

where $p_0 = y^{-\alpha}$,

$$p_{2k-1} = (2k-1)\int_{y_0}^{y} p_{2k-2}\,dt, \qquad p_{2k} = 2ky^{-\alpha}\int_{y_0}^{y} t^{\alpha-1}p_{2k-1}\,dt, \qquad q_0 = 1,$$

$$q_{2k-1} = (2k-1)y^{-\alpha}\int_{y_0}^{y} q_{2k-2}t^{\alpha-1}\,dt, \qquad q_{2k} = 2k\int_{y_0}^{y} q_{2k-1}\,dt.$$

For arbitrary real numbers α_k and β_k, the functions V_k are solutions of equation (3_0) and have at the point $(0, y_0)$ a zero of order k. Let Ψ_k be the solution of equation (7) for $\mu = \mathrm{Im}(\varphi V_k)$ on Γ. Then the general solution Φ of equation (3_0) under the homogeneous conditions corresponding to (4) can be represented in the form

$$\Phi = \sum_{k=0}^{n-1}(V_k - T(\Psi_k)) + \Phi_n,$$

where Φ_n is a solution of equation (3_0) under the homogeneous conditions corresponding to (4), and having a zero of order n at the point $(0, y_0)$. The latter problem has one linearly independent solution. From the linear independence of the functions $\Phi_n,\ V_k - T(\Psi_k)$ $(k = 0, 1, \dots, n-1)$, the validity of our assertion follows.

Let now $n < 0$. Introduce the new unknown function $w_0 = T^{-1}(w)$. In order that w be a solution of the problem, it is necessary and sufficient that w_0 have at the point $(0, y_0)$ a zero of order $-n$ and satisfy the integral equation

$$w_0 + K(A_1 w_0 + B_1 w_0) = K(F_0) + \Omega(f) + iC, \tag{9}$$

where C is an arbitrary constant, while A_1, B_1, F_0 are determined uniquely through A, B, φ, F. Equation (9) is always solvable. In order that the solution of equation (9) vanish at the point $(0, y_0)$ to the required order, it is necessary to impose $-2n$ conditions on F, f, and C. The fulfillment of one of them can be achieved by the choice of the arbitrary C. It is not difficult to prove that these conditions have the form (5).

Remark. Theorems analogous to 1-5 also hold for the system

$$u_x - v_y + au + bv = f_1,$$

$$y^m u_y + v_x + cu + dv = f_2,$$

where $1 < m < 2,\ c = ay^{m-1} + o(y^{m-1})$.

For a system of the form

$$y^m u_x - v_y + au + bv = f_1,$$

$$u_y + v_x + cu + dv = f_2$$

Theorem 1 always holds for any $m > 0$.

Institute of Mathematics with Computing Center
Siberian Branch of the Academy of Sciences of the USSR

Received
18 XI 1963

CITED LITERATURE

1. A. V. Bitsadze, *Equations of Mixed Type*, Publishing House of the Academy of Sciences of the USSR, 1959.

2. I. I. Danilyuk, *Siberian Mathematical Journal*, 4, issue 6 (1963).

3. A. Weinstein, *Trans. Am. Math. Soc.*, 63, No. 2 (1948).

4. L. Bers, A. Gelbart, *Quart. Appl. Math.*, 1, No. 2 (1943).

Note: Figure translations are in progress. See original paper for figures.