CYBERNETICS AND CONTROL THEORY

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Abstract

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ON THE OPTIMAL LINEARIZATION OF A NONLINEAR CONTROL OBJECT WITH UNKNOWN PARAMETERS BY A LINEAR SYSTEM WITH CONSTANT PARAMETERS IN THE ABSENCE OF DISTURBANCES

In many cases it is necessary to solve problems of controlling objects whose parameters are unknown or difficult to determine. For cases in which it is possible to determine the output signals for a prescribed family of input signals, optimal linearization in the absence of disturbances is of practical interest. Suppose we have a nonlinear control object with one input and one output and with an unknown mathematical description, i.e., it is a "black box." We consider a continuous input signal $x(t)$ and an output signal $y(t)$, which are deterministic (nonrandom) functions of time t; the object is initially unexcited, i.e., at least $y(0) = 0$. There may be available one realization $\langle x(t), y(t)\rangle$ or several realizations $\langle x_i(t), y_i(t)\rangle$, $i = 1, 2, \dots, N$, $t \in [0, T]$, where T is a prescribed number. Linearization by one realization will be called **partial linearization**; it is of particular interest for small changes of the input signal. Then optimal linearization (as in the case of disturbances) is understood in the sense of ensuring a minimum of the mean square error ε_{cp}, equal to

$$\varepsilon_{\text{cp}} = \frac{1}{T}\int_0^T [y(t) - \tilde{y}(t)]^2\,dt = \frac{1}{T}\int_0^T \left[y(t) - \int_0^t k(t-\tau)x(\tau)\,d\tau\right]^2 dt, \qquad (1)$$

where $\tilde{y}(t)$ is the output signal of a fictitious initially unexcited linear object (1); $k(t)$, $t \geq 0$, is its impulse response function; $x(0) \neq 0$ and accordingly $k(0) = 0$.

The optimal impulse response function $k_{\text{opt}}(t)$ in the present case ensures $\varepsilon_{\text{cp}} = 0$ and satisfies the Volterra integral equation of the first kind

$$\int_0^t x(t-\xi)k_{\text{opt}}(\xi)\,d\xi = y(t), \qquad t \geq 0, \qquad (2)$$

with difference kernel $x(t-\xi)$, defined in the region $G\{0 \le \xi \le t \le T\}$. In practice, the functions $x(t)$ and $y(t)$ are often continuously differentiable, and for some forms it is possible to obtain an exact solution of equation (2). In most cases the sought $k_{\text{opt}}(t)$ can be found with any degree of accuracy by the method of series in the form of the approximate value

$$\tilde{k}_{\text{opt}}(t) = \sum_{i=1}^{l} \frac{k^{(i)}(0)}{i!}\, t^i, \qquad t \ge 0, \tag{3}$$

where

$$k'(0) = \frac{y''(0)}{x(0)}, \dots, \qquad k^{(l)}(0) = \frac{y^{(l+1)}(0) - \sum\limits_{i=1}^{l-1} k^{(i)}(0)x^{(l-i)}(0)}{x(0)}. \tag{4}$$

This is easily obtained by successive differentiation of (2) with respect to t, taking $t = 0$. Let the integer $r \geqslant 1$ be the first number for which $k^{(r)}(0) = 0$. Then the order n of the fictitious linear system with constant parameters is equal to $n = r + 2$. The error is

$$\varepsilon(t) = k_{\text{opt}}(t) - \tilde{k}_{\text{opt}}(t) = \frac{k^{(l+1)}(\eta)}{(l+1)!} t^{l+1},$$

$0 < \eta < t$, and can be made negligibly small by choosing large l, provided the derivatives are bounded. The corresponding "transfer" function will be:

$$\widetilde{K}(p) = \int_0^\infty e^{-pt}\tilde{k}_{\text{opt}}(t)\, dt = \sum_{i=1}^{l} \frac{k^{(i)}(0)}{p^{i+1}}. \tag{5}$$

If there are several realizations of the input and output signals, then for each of them one can find $k_{\text{opt},i}(t)$ as above and choose as $k_{\text{opt}}(t)$ the quantity

$$\overline{k}_{\text{opt}}(t) = \frac{1}{N}\sum_{i=1}^{N} k_{\text{opt},i}(t). \tag{6}$$

One can also apply another method of using several realizations. Let us find the mean values of the input and output signals over the realizations

$$\overline{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \qquad \overline{y}(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t) \tag{7}$$

and solve the following Volterra integral equation:

$$\int_0^t \overline{x}(t-\xi)k_{\text{opt}}(\xi)\, d\xi = \overline{y}(t), \qquad t \geqslant 0, \tag{8}$$

which determines a somewhat different impulse response function $k_{\text{opt}}(t)$, referred to the mean values of the realizations.

Let us note that, instead of solving the Volterra equation (2), one may use another method—the Ritz method of minimizing ε_{av}, especially when $y(t)$ is discontinuous (necessarily), and also when $x(t)$ and $y(t)$ are not continuously differentiable and when $x(0) = 0$. For this purpose we approximate the continuous function $k_{\text{opt}}(t)$, $t \in [0, T]$, by a "generalized polynomial"

$$\tilde{k}_{\text{opt}}(t) = \sum_{i=0}^{l} a_i \omega_i(t),$$

where $a_0, a_1, \dots, a_l$ are constant coefficients to be determined, and $\omega_i(t)$, $i = 0, 1, \dots, l$, is a system of chosen linearly independent basis functions. Then

$$\varepsilon_{\text{av}} = \frac{1}{T} \int_0^T \left[y(t) - \sum_{i=0}^{l} a_i f_i(t) \right]^2 dt, \tag{9}$$

where

$$f_i(t) = \int_0^t x(t-\tau)\omega_i(\tau)\, d\tau, \qquad t \in [0, T]. \tag{10}$$

The coefficients a_i that ensure the minimum of ε_{av} are found from the system of linear algebraic equations

$$\sum_{i=1}^{l} b_{ij} a_i = c_j, \qquad j = 0, 1, \dots, l, \tag{11}$$

obtained from the conditions

$$\frac{\partial \varepsilon_{\text{av}}}{\partial a_i} = 0 \qquad (i = 0, 1, \dots, l),$$

where

$$b_{ij} = \int_0^T f_i(t) f_j(t)\, dt, \qquad i, j = 0, 1, \dots, l,$$

$$c_j = \int_0^T y(t) f_i(t)\, dt, \qquad j = 0, 1, \dots, l. \tag{12}$$

The solution of system (11) is given by Cramer's formulas: $a_i = D_i / D$, assuming, of course, that the determinant D of the system is nonzero.

Thus, it has been shown that the optimization problem in the absence of disturbances can be solved on the basis of the solution of a linear Volterra integral equation of the first kind or by the Ritz method.

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REFERENCES

1. A. V. Solodov, *Linear Systems of Automatic Control with Variable Parameters*, Moscow, 1962, p. 32.

Note: Figure translations are in progress. See original paper for figures.