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Abstract

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MATHEMATICS

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A Method for Finding Green' s Functions of Boundary-Value Problems for the Heat-Conduction Equation for a Line Segment with Uniformly Moving Boundaries

(Presented by Academician A. A. Dorodnitsyn, 4 II 1964)

It is known that the solution of boundary-value problems described by the heat-conduction equation for domains with moving boundaries leads to a system of Volterra integral equations of the second kind ^(1,2). Owing to the difficulty of solving this system, only the simplest problems have so far been considered ⁽³⁻⁵⁾.

The difficulty of solving the problems under consideration depends on the nature of the domain and on the laws of motion of the boundaries, and is ultimately determined by the dependence of the characteristic size of the domain on time. In this article a general method is given for solving boundary-value problems for a segment of a straight line with uniformly moving boundaries. The method is applied for the purpose of determining Green' s functions, but it can also be used for the direct solution of a given boundary-value problem. To determine Green' s functions, the heat-conduction equation is written in the form

$$\left(a \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t} \right) G(z, t; z_0, t_0) = \frac{1}{2} \text{sign}(z_0 - z) \delta(z - z_0) \delta(t - t_0), \quad (1)$$

somewhat different from the traditional one, which is due to the fact that the methods used are analytical and are based on the direct application of operational calculus to this equation.

Because of lack of space we shall consider, as an example, the simplest problem, when the required Green' s function must satisfy boundary conditions of the first kind:

$$G(z, t; z_0, t_0)|_{z=v_1 t} = 0, \quad (2)$$

$$G(z, t; z_0, t_0)|_{z=l+v_2 t} = 0. \quad (3)$$

We introduce a coordinate system moving together with the left boundary of the line, for which we put $\xi = z - v_1 t$; in addition, we put $\xi_0 = z_0 - v_1 t_0$. The general solution of equation (1), written in the moving coordinate system, in the space of Laplace transforms depends on two arbitrary functions $A(s)$ and $B(s)$, determined from the boundary conditions. Using boundary condition (2), we find that

$$B(s) = -A(s) + \frac{1}{2q} \exp\left(\frac{v_1}{2a} \xi_0 - st_0\right) \left(\exp\left(\frac{q}{2a} \xi_0\right) - \exp\left(-\frac{q}{2a} \xi_0\right) \right)$$

$$(q = \sqrt{v_1^2 + 4as}).$$

Substituting the value of the function $B(s)$ in its place, we pass to the original by means of the Riemann-Mellin inversion formula. Requiring fulfillment of boundary condition (3), we arrive at the integral equation

with respect to the function $A(s)$:

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left\{ \left[A(s) - \frac{1}{2q} \exp\left(\frac{v_1 - q}{2a} \xi_0 - st_0\right) \right] \exp\left[\frac{-v_1 + q}{2a}(l + v_0 t)\right] \right.$$

$$\left. + \left[-A(s) + \frac{1}{2q} \exp\left(\frac{v_1}{2a} \xi_0 - st_0\right) \left(2 \exp\left(\frac{q}{2a} \xi_0\right) - \exp\left(-\frac{q}{2a} \xi_0\right) \right) \right] \exp\left[\frac{-v_1 - q}{2a}(l + v_0 t)\right] \right\} \exp(st) ds = 0 \quad (4)$$

To solve this integral equation we again use the Laplace transform. We multiply both sides of the equation by e^{-pt} , where $\operatorname{Re} p > \nu > \sigma_0 > 0$ (here ν is $\max[\operatorname{Re}(s + \frac{-v_1 + q}{2a} v_0), \operatorname{Re}(s + \frac{-v_1 - q}{2a} v_0)]$), and integrate with respect to t from 0 to ∞ . Since, by absolute convergence, the order of integration may be changed, after carrying out the integration with respect to t we have

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left\{ \left[A(s) - \frac{1}{2q} \exp\left(\frac{v_1 - q}{2a} \xi_0 - st_0\right) \right] \frac{\exp\left(\frac{-v_1 + q}{2a} l\right)}{p - s - \frac{-v_1 + q}{2a} v_0} \right.$$

$$\left. + \left[-A(s) + \frac{1}{2q} \exp\left(\frac{v_1}{2a} \xi_0 - st_0\right) \left(2 \exp\left(\frac{q}{2a} \xi_0\right) - \exp\left(-\frac{q}{2a} \xi_0\right) \right) \right] \frac{\exp\left(\frac{-v_1 - q}{2a} l\right)}{p - s - \frac{-v_1 - q}{2a} v_0} \right\} ds = 0. \quad (5)$$

Consider the closed contour of integration made up of the line σ_0 and the arc of the circle C_R situated to the right of this line. It is known that if s tends to infinity in such a way that $\operatorname{Re} s$ increases without bound, then the transform $F(s)$ of the original function $f(t)$ tends to zero. Then, as $R \rightarrow \infty$,

$$\int_{C_R} F_i(s) \frac{ds}{p-s-\frac{-v_1 \pm q}{2a} v_0} \rightarrow 0.$$

Consequently, the line of integration in (5) may be replaced by the indicated closed contour, for which $R \rightarrow \infty$. Applying the residue theorem to (5), we obtain the linear inhomogeneous functional equation

$$\begin{aligned} & \left\{ 2(2ar - v_0)A \left[p + v_0 \left(\frac{v_2}{2a} - r \right) \right] - \exp \left[- \left(p + v_0 \left(\frac{v_2}{2a} - r \right) \right) t_0 + \frac{v_1 + 2ar - v_0}{2a} \xi_0 \right] \right\} \exp \left(\frac{2ar - v_0}{2a} l \right) + \\ & + \left\{ -2(2ar + v_0)A \left[p + v_0 \left(\frac{v_2}{2a} + r \right) \right] - \exp \left[- \left(p + v_0 \left(\frac{v_2}{2a} + r \right) \right) t_0 + \frac{v_1 - 2ar - v_0}{2a} \xi_0 \right] \right\} \\ & + 2 \exp \left[- \left(p + v_0 \left(\frac{v_2}{2a} + r \right) \right) t_0 + \frac{v_1 + 2ar + v_0}{2a} \xi_0 \right] \exp \left(- \frac{2ar + v_0}{2a} l \right) = 0, \end{aligned} \quad (6)$$

whose kernel can be reduced to a difference kernel; for this it is necessary to set

$$p + v_0 \left(\frac{v_2}{2a} - r \right) = x^2 - \frac{v_1^2}{4a} \quad \left(r = \sqrt{\frac{p}{a} + \frac{v_2^2}{4a^2}} \right).$$

Denoting $2x\sqrt{a}A\left(x^2 - \frac{v_1^2}{4a}\right)$ by $F(x)$, we arrive at a finite-difference equation of the form

$$\begin{aligned} & F \left(x + \frac{v_0}{\sqrt{a}} \right) - \exp \left[l \left(2 \frac{x}{\sqrt{a}} + \frac{v_0}{a} \right) \right] F(x) \\ & = \frac{1}{2} \exp \left(\frac{v_1}{2a} \xi_0 + \frac{v_1^2}{4a} t_0 - x^2 t_0 \right) \left\{ 2 \exp \left[- \frac{2xv_0}{\sqrt{a}} t_0 + \xi_0 \left(\frac{x}{\sqrt{a}} + \frac{v_0}{a} \right) - \frac{v_0^2}{a} t_0 \right] \right. \\ & \quad \left. - \exp \left[- \frac{2xv_0}{\sqrt{a}} t_0 - \xi_0 \left(\frac{x}{\sqrt{a}} + \frac{v_0}{a} \right) - \frac{v_0^2}{a} t_0 \right] - \exp \left[\frac{x}{\sqrt{a}} (2l - \xi_0) + \frac{v_0 l}{a} \right] \right\}. \end{aligned} \quad (7)$$

In the case $v_0 = 0$, the solution of equation (7) presents no difficulty. If, however, $v_0 \neq 0$, then an equation of the type

$$F(x + \alpha) - be^{\beta x} F(x) = c(x)$$

by means of the substitution $F(x) = \exp\left(\frac{\beta}{2\alpha}x^2\right) f(x)$ is reduced to an equation with constant coefficients

$$\frac{1}{b} \exp\left(\frac{\alpha\beta}{2}\right) f(x + \alpha) - f(x) = \frac{1}{b} \exp\left(-\frac{\beta}{2\alpha}x^2 - \beta x\right) c(x).$$

At the same time, by direct verification one may make sure that if the series $-\sum_{k=0}^{\infty} a^k g(x + k\alpha)$ converges, then the function represented by this series is the desired particular solution of the equation

$$af(x + \alpha) - f(x) = g(x).$$

But a series of the type $\sum_{k=0}^{\infty} \exp\left[-\frac{\alpha\beta}{2}(k + \rho)^2\right]$, to which the case under consideration leads, converges if $\alpha\beta > 0$, which corresponds to $v_0 > 0$. Then the solution of the functional equation (6) can be represented in the form

$$\begin{aligned} F(x) = & \exp\left(\frac{v_1}{2a}\xi_0 + \frac{v_1^2}{4a}t_0 - x^2t_0\right) \left(\frac{1}{2} \exp\left[-\frac{x\xi_0}{\sqrt{a}}\right] \right. \\ & + \sum_{k=1}^{\infty} \exp\left(-k^2\frac{l_0v_0}{a}\right) \left\{ \exp\left[-\frac{kv_0}{a}\xi_0 - \frac{x}{\sqrt{a}}(2l_0k + \xi_0)\right] \right. \\ & \left. \left. - \exp\left[\frac{kv_0}{a}\xi_0 - \frac{x}{\sqrt{a}}(2l_0k - \xi_0)\right] \right\} \right), \quad (l_0 = l + v_0t_0). \end{aligned} \quad (8)$$

Substituting its value for the function $A(s)$ and passing from the transform space to the space of originals, we obtain the following value for the Green' s function:

$$\begin{aligned} G(\xi, t; \xi_0, t_0) = & \frac{1}{2\sqrt{\pi a(t - t_0)}} \exp\left[\frac{v_1}{2a}(\xi_0 - \xi) - \frac{v_1^2}{4a}(t - t_0)\right] \times \\ & \times \sum_{-\infty}^{\infty} \exp\left(-k^2\frac{l_0v_0}{a} - k\frac{v_0\xi_0}{a}\right) \mu(2l_0k + \xi_0), \end{aligned} \quad (9)$$

where

$$\mu(\xi_0) = \exp\left[-\frac{(\xi_0 - \xi)^2}{4a(t - t_0)}\right] - \exp\left[-\frac{(\xi_0 + \xi)^2}{4a(t - t_0)}\right].$$

Examining the solution obtained, it is not hard to see that, because of the presence of the factor $\exp\left[-\frac{l_0^2k^2}{a(t - t_0)}\right]$ in the function $\mu(2l_0k + \xi_0)$, it also

converges for negative values of v_0 , if $t_0 < t < -l/v_0$. But for $t = -l/v_0$ the prescribed domain disappears. Consequently, the solution is valid for arbitrary values of v_0 .

For other boundary conditions, the specific features of the method consist only in solving the finite-difference equation and passing to the original.

In conclusion we note that this method can also be extended to certain other domains and laws of motion of the boundaries.

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