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Abstract

Full Text

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On the Stability of Certain Computational Processes

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Let X_n and Y_n, $n = 1, 2, ...$, be Banach spaces and let A_n be linear operators acting from X_n into Y_n, where for every n the operator A_n^{-1} exists and is defined on the entire space Y_n. Consider the computational process consisting in solving the sequence of equations

$$A_n x^{(n)} = y^{(n)}. \tag{1}$$

Alongside equations (1), consider the sequence of equations

$$(A_n + \Gamma_n) z^{(n)} = y^{(n)} + \delta^{(n)}, \tag{2}$$

where the operators Γ_n are also linear.

We shall say that the computational process (1) is **stable** (with respect to changes in the operator A_n and the free term $y^{(n)}$) if there exist constants p, q, r, independent of n, such that for $\|\Gamma_n\| \le r$ and for arbitrary $\delta^{(n)}$, equations (2) are solvable and the inequality holds $\eta^{(n)} = z^{(n)} - x^{(n)}$

$$\|\eta^{(n)}\| \le p\|\Gamma_n\| + q\|\delta^{(n)}\|. \tag{3}$$

Theorem 1. *For the stability of the computational process (1), it is necessary and sufficient that, independently of n, the norms of the operators A_n^{-1} and the elements $A_n^{-1} B_{nA} n^{-1} y^{(n)} = A_n^{-1} B_{nx}^{(n)}$ be bounded, where B_n is an arbitrary operator of unit norm acting from X_n into Y_n.*

Necessity. 1) Let $\Gamma_n = 0$. Then $A_n \eta^{(n)} = \delta^{(n)}$ and $\|\eta^{(n)}\| \le q\|\delta^{(n)}\|$. Hence $\|A_n^{-1}\| \le q$.

2) Let now $\delta^{(n)} = 0$ and $\Gamma_n = \varepsilon B_n$, where $\|B_n\| = 1$ and ε is a constant, $0 < \varepsilon < r$. In this case $\eta^{(n)}$ satisfies the equation

$$\eta^{(n)} + \varepsilon A_n^{-1} B_n \eta^{(n)} = -\varepsilon A_n^{-1} B_{nx}^{(n)}.$$

Hence

$$\|\eta^{(n)}\| \geq \frac{\varepsilon \|A_n^{-1} B_{nx}^{(n)}\|}{1+\varepsilon q}.$$

Comparing this with inequality (3), which in the present case has the form $\|\eta^{(n)}\| \leq p\varepsilon$, we find that $\|A_n^{-1} B_{nx}^{(n)}\| \leq p(1+\varepsilon q)$. Since ε is arbitrarily small, $\|A_n^{-1} B_{nx}^{(n)}\| \leq p$.

Sufficiency. Let $\|A_n^{-1}\| \leq C_1$, $\|A_n^{-1} B_{nx}^{(n)}\| \leq C_2$, where the constants C_1 and C_2 do not depend on n. Put $r = \beta C_1^{-1}$, where β is a constant, $0 < \beta < 1$, and require that $\|\Gamma_n\| \leq r$. Form the equations

$$(A_n + \Gamma_n) z_1^{(n)} = y^{(n)}, \qquad (A_n + \Gamma_n) z_2^{(n)} = \delta^{(n)};$$

then $\eta^{(n)} = (z_1^{(n)} - x^{(n)}) + z_2^{(n)}$. It is easy to see that

$$\|z_2^{(n)}\| \leq \|(I_n + A_n^{-1}\Gamma_n)^{-1}\| \, \|\delta^{(n)}\| C_1 \leq \frac{C_1}{1-\beta} \|\delta^{(n)}\|;$$

here I_n is the identity operator.

Further, putting $\Gamma_n = \|\Gamma_n\| B_n$, $\|B_n\| = 1$, we have

$$z_1^{(n)} - x^{(n)} = -\|\Gamma_n\| (I_n + A_n^{-1}\Gamma_n)^{-1} A_n^{-1} B_{nx}^{(n)}$$

and, consequently,

$$\|z_1^{(n)} - x^{(n)}\| \leq \frac{\|\Gamma_n\|}{1-\beta} \|A_n^{-1} B_{nx}^{(n)}\| \frac{C_2}{1-\beta} \|\Gamma_n\|;$$

inequality (3) is thus satisfied for the following values of the constants

$$p = \frac{C_2}{1-\beta}, \qquad q = \frac{C_1}{1-\beta}, \qquad r = \frac{\beta}{C_1}.$$

If the quantity $\|x^{(n)}\| = \|A_n^{-1} y^{(n)}\|$ is bounded, then for the stability of the computational process (1) it is necessary and sufficient that the norms $\|A_n^{-1}\|$ be bounded. The quantity $\|x^{(n)}\|$ will be bounded if, for example, X_n, $n = 1, 2, \ldots$, are subspaces of some Banach space X, and in this space $\lim x^{(n)}$ exists.

Theorem 2. *If the spaces X_n are Hilbert spaces and the norms $\|A_n^{-1}\|$ are bounded below by a positive constant independently of n, then for the stability of process (1) it is necessary and sufficient that the quantities $\|A_n^{-1}\|$ and $\|x^{(n)}\| = \|A_n^{-1} y^{(n)}\|$ be bounded above independently of n.*

Theorem 2 will be proved if we establish that from the conditions

$$\|A_n^{-1}B_{nx}^{(n)}\| \le C_2, \qquad \|A_n^{-1}\| \ge C_3,$$

where C_2, C_3 are positive constants, there follows the boundedness of the norms $\|x^{(n)}\|$. Suppose the contrary, and let $\|x^{(n_k)}\| \to \infty$. Choose an element $t^{(n_k)} \in Y_{n_k}$ so that $\|t^{(n_k)}\| = 1$ and $\|A_{n_k}^{-1}t^{(n_k)}\| \ge \frac{1}{2}\|A_{n_k}^{-1}\|$. Construct an operator B_{n_k}, $\|B_{n_k}\| = 1$, which, on elements of the form $\lambda x^{(n_k)}$, acts according to the formula

$$B_{n_k}(\lambda x^{(n_k)}) = \lambda\|x^{(n_k)}\|t^{(n_k)},$$

and which annihilates the elements orthogonal to $x^{(n_k)}$. Then

$$\|A_{n_k}^{-1}B_{n_k}x^{(n_k)}\| \ge \frac{1}{2}\|A_{n_k}^{-1}\|\,\|x^{(n_k)}\| \to \infty,$$

contrary to the condition.

Let us give several examples.*

Example 1. Let A be a positive definite operator acting in a Hilbert space H, and suppose the problem of minimizing the functional

$$F(u) = \|u\|_A^2 - (u, f) - (f, u) \tag{4}$$

is solved by the Ritz process. Let $\varphi_k,\ k = 1, 2, \dots$, be coordinate elements subject to the usual conditions ([1]), and

$$u_n = \sum_{k=1}^{n} a_k^{(n)}\varphi_k, \tag{5}$$

the Ritz approximate solution of the above variational problem. The vector $x^{(n)} = (a_1^{(n)}, a_2^{(n)}, \dots, a_n^{(n)})'$ is determined from the equation

$$R_{nx}^{(n)} = y^{(n)}, \tag{6}$$

where R_n is the matrix $\|[\varphi_k, \varphi_j]_A\|_{j,k=1}^{j,k=n}$; $y^{(n)} = (f_1, f_2, \dots, f_n)'$; $f_k = (f, \varphi_k)$.

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* On the terminology and notation, see [1].

Equation (6) falls under type (1) if one sets $X_n = Y_n, E_n$, where E_n is an n-dimensional unitary space, and $A_n = R_n$. It is not difficult to prove that $\|R_n^{-1}\| \ge |\varphi_1|_A^2$, and then it follows from Theorem 2 that the process of computing the Ritz coefficients from equations (6) is stable if and only if the coordinate system is strongly minimal [2] in the space H_A, i.e., if $\lambda_1^{(n)} \geq \lambda_0 = \text{const} > 0$, where $\lambda_1^{(n)}$ is the smallest eigenvalue of the matrix R_n; the sufficiency of this condition was proved in [3].

Example 2. Let $Y_n = E_n$, and let X_n be the subspace of the space H_A that is the linear span of the elements $\varphi_1, \varphi_2, \dots, \varphi_n$. The approximate solution u_n of the variational problem (4) is determined from the equation

$$(S_n^*)^{-1} u_n = y^{(n)}, \tag{7}$$

where S_n is the operator which assigns to the element u_n (formula (5)) the vector $x^{(n)}$. Since u_n tends in the metric of H_A to the limit (1), the norms $|u_n|_A$ are bounded in the aggregate, and for the stability of the process (7) it is necessary and sufficient that the quantities $\|S_n\|$ be bounded in the aggregate. The relation $(S_n^*)^{-1} S_n^{-1} = R_n$ holds; hence $\|S_n\| = [\lambda_1^{(n)}]^{-1/2}$, and the necessary and sufficient condition for stability of the process (7) reduces to the strong minimality of the coordinate system in the space H_A. By definition, the inequality characterizing the stability of the named process has the form

$$|\eta_n|_A \leq p\|\Gamma_n\| + q\|\delta^{(n)}\|,$$

where Γ_n and $\delta^{(n)}$ may be regarded as the errors of the operator $(S_n^*)^{-1}$ and of the vector $y^{(n)}$, respectively.

Suppose that these errors arise only as a result of inexact computation of the scalar products (f, φ_k) and the energy products $[\varphi_k, \varphi_j]_A$. Then the operator S_n^{-1} is free of error, as is seen from the formula

$$S_n^{-1} x^{(n)} = u_n = \sum_{k=1}^{n} a_k^{(n)} \varphi_k.$$

Denoting by $\overline{\Gamma}_n$ the error of the matrix R_n, we have $\Gamma_n = \overline{\Gamma}_n S_n$. Hence $\|\Gamma_n\| \leq \|\overline{\Gamma}_n\| [\lambda_1^{(n)}]^{-1/2}$. If the coordinate system is strongly minimal in H_A, then, putting $p\lambda_0^{-1/2} = p_1$, $q = q_1$, we obtain the inequality established in [4]

$$|\eta_n|_A \leq p_1\|\overline{\Gamma}_n\| + q_1\|\delta^{(n)}\|.$$

3. Let the operator A be defined as above, and let the operator K be such that the product $A^{-1}K$ can be extended to an operator that is completely continuous in the space H_A. From Theorem 1 follow the theorems of the article [4], by virtue of which the processes of determining the vector of coefficients and the approximate solution in the Bubnov-Galerkin method for the equation

$$(A + K)u = f$$

are stable if the coordinate system is strongly minimal in the space H_A.

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Note: Figure translations are in progress. See original paper for figures.