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Abstract

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A GENERALIZATION OF A THEOREM OF P. S. ALEXANDROV ON ONE-POINT BICOMPACTIFICATION

(Presented by Academician P. S. Alexandrov, 11 II 1964)

The well-known theorem of P. S. Alexandrov on one-point bicompactification ([1]), p. 390, Theorem 9, is generalized to T_1-spaces for noncontractible bicompact extensions ([2]), Definition 1. I give this definition.

Definition*. A bicompact T_1-extension bX of a space X is called **noncontractible** if for no closed set $F \subset X$ does there exist a bicompact space $\Phi \subset bX$ such that

$$[F]_{bX} \supset \Phi \supset F; \qquad \Phi \neq [F]_{bX}.$$

Theorem. *Every non-bicompact T_1-space X can be noncontractibly bicompactified in a unique way, up to homeomorphism, by adjoining one point $\xi \notin X$.*

Proof. Let X be a T_1-space. Put

$$\alpha X = X \cup \xi, \qquad \xi \notin X.$$

Define a topology in αX as follows: the open sets in αX not containing the point ξ are all the open sets in X, and only them, while the open sets in αX containing the point ξ are the sets of the form

$$\xi \cup (X \setminus \Phi),$$

where $\Phi \subset X$ is a closed bicompact set. It is easily verified that the axioms of a topological space are satisfied in αX.

The space αX is bicompact. Indeed, let $\{U_\alpha\}$ be an arbitrary open covering of the space αX. From $\{U_\alpha\}$ one can choose a finite subcovering of the space αX, because among the sets U_α there is some set

$$U_0 = U_0\xi = \xi \cup (X \setminus \Phi),$$

and from the remaining sets of the system $\{U_\alpha\}$, which cover at least Φ, one can choose a finite system of sets

$$U_1, U_2, \dots, U_s,$$

covering Φ, since Φ is bicompact. Then the sets

$$U_0, U_1, U_2, \dots, U_s$$

together cover the whole space αX. Thus the bicompactness of the space αX is proved. It is obvious that αX is an extension of the space X.

Let us prove the noncontractibility of αX. Let F be an arbitrary closed set of the space X. If F is not bicompact, then, by the definition of the topology in αX,

$$[F]_{\alpha X} = F \cup \xi.$$

Obviously, if the space $b'F$ is bicompact and

$$[F]_{\alpha X} \supset b'F \supset F,$$

then

$$b'F = F \cup \xi,$$

and, consequently,

$$[F]_{\alpha X} = b'F.$$

If, however, F is bicompact, then

$$[F]_{\alpha X} = F,$$

since $\xi \cup (X \setminus F)$ is a neighborhood of the point ξ not intersecting F. Therefore, in accordance with the definition, αX is noncontractible.

Uniqueness. Let αX and $\alpha' X$ be two one-point noncontractible bicompact extensions of the space X. We shall prove that they are homeomorphic. Denote by τ and τ' the topologies in αX and, respectively, in $\alpha' X$. Take an arbitrary $U \in \tau$ for which $\xi \in U$. Then from the definition of the topologies τ and τ' it follows that $U \in \tau'$. Now take any such

$$V \in \tau, \qquad V \supset \xi.$$

Then the set $\alpha X \setminus \Phi$ is closed, $\Phi \subset X$.

By the noncontractibility of the bicompact extension αX, the set Φ is bicompact. Hence the set $V \in \tau'$, consequently,

$$\tau \subset \tau'.$$

One may interchange the roles of αX and $\alpha' X$; therefore

$$\tau \supset \tau'.$$

Thus we have proved that

$$\tau = \tau',$$

i.e. αX and $\alpha' X$ are homeomorphic. The theorem is proved.

* Introduced by A. V. Arhangel' skii.

In the case of locally bicompact Hausdorff spaces, this unique noncontractible bicompactification by one point coincides with the well-known bicompactification by one point of P. S. Aleksandrov.

For valuable advice in carrying out this work, I express my sincere gratitude to A. V. Arhangel' skii.

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CITED LITERATURE

1. P. S. Aleksandrov, *Introduction to the General Theory of Sets and Functions*, Moscow—Leningrad, 1948.

2. P. K. Osmatesku, *Vestnik Moskovskogo universiteta*, ser. mathematics and mechanics, No. 6, 45 (1963).

Note: Figure translations are in progress. See original paper for figures.