



Soviet-era science, translated into English

Aerodynamics

Corresponding Member of the Academy of Sciences of the USSR V.
V. Struminskii

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.39081>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Aerodynamics

Corresponding Member of the Academy of Sciences of the USSR V. V. Struminskii

ON A METHOD FOR SOLVING THE BOLTZMANN KINETIC EQUATION

For the study of many problems of modern aerodynamics that are of great practical importance, the Boltzmann kinetic equation has in recent years been successfully applied. Several different methods for solving this equation are known, in particular, the method of successive approximations proposed by Hilbert ⁽¹⁾, and the modified method of successive approximations proposed by Enskog ⁽²⁾. In the author's work ⁽³⁾ it was shown that the range of applicability of Hilbert's method—the ordinary small-parameter method—is limited. To extend the possibilities of the small-parameter method as applied to problems of the nonlinear theory of oscillations, Poincaré ⁽⁴⁾ proposed, along with the unknown function, representing also the period of oscillation T as a power series in the small parameter. Poincaré's method was further developed and generalized in the works of Lighthill ⁽⁵⁾, who, in solving a number of aerodynamic problems, along with the unknown function represented the coordinates x, y, z ⁽⁶⁾ as power series as well.

Poincaré's method also permits further generalization. In particular, when solving certain nonperiodic problems it proves advisable, along with the unknown function, to represent the time t , or certain functions of t , as a power series in the small parameter. Such a method was applied by the author in studying the character of the development of aerodynamic disturbances ⁽⁷⁾.

In the present work, precisely such a modification of Poincaré's method is applied to the study of solutions of the Boltzmann equations. The paper shows that this method has no limitations in solving the Boltzmann equation and is suitable for all approximations. Enskog's method is a special case of the more general method applied here for solving the Boltzmann equation.

Let us write the Boltzmann equation in the following abbreviated form ⁽³⁾:

$$J(F : F) = D(F). \quad (1)$$

We shall seek such solutions of equation (1) as lead to distribution functions of normal structure (the first moments are determined by the first terms of the expansion of the initial function in a series in the small parameter). We represent the distribution function in the form of Hilbert's power series

$$F(r, v, t) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^n F^{(n)}(r, v, t), \quad (2)$$

where λ is a small parameter.

Suppose that for the series (2) the conditions are strictly satisfied:

$$\begin{aligned} \rho &= \iiint F dv = \iiint \frac{F^{(0)}}{\lambda} dv, & \rho U &= \iiint v F dv = \iiint \frac{F^{(0)}}{\lambda} dv, \\ 3P &= \iiint (v - U)^2 F dv = \iiint (v - U)^2 \frac{F^{(0)}}{\lambda} dv. \end{aligned} \quad (3)$$

As was shown in ⁽³⁾, the series (2) cannot be used to solve the Boltzmann equation, because in this case the solvability conditions will not be satisfied. To satisfy these conditions it is necessary in each—

in the second approximation introduce (in a manner different from Hilbert' s method) a set of 5 arbitrary functions, which can be determined from 5 solvability conditions. To this end let us assume that the distribution function, in addition to its explicit dependence on time, coordinates, and momenta (velocities), also depends on 5 as yet unknown parameters, which in turn depend on time. As these parameters one may take the set of 5 first moments (3), on which, in particular, the distribution function of the zeroth approximation depends ⁽³⁾.

Let us represent the time dependence of the distribution function in the form

$$F(r, v, t) = F(r, v, \rho(r, t), U(r, t), P(r, t), \tau(t)). \quad (4)$$

With respect to the set of as yet unknown functions $\rho(r, t)$, $U(r, t)$, $P(r, t)$, let us assume that they satisfy the system of equations

$$\frac{\partial \rho}{\partial t} = \sum_{n=0}^{\infty} \lambda^n A_n(\rho, U, P), \quad \frac{\partial U}{\partial t} = \sum_{n=0}^{\infty} \lambda^n B_n(\rho, U, P), \quad \frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} \lambda^n C_n(\rho, U, P), \quad (5)$$

where the functions A_n, B_n, C_n are to be determined. The unknown function $\tau(t)$ is determined from the condition

$$\frac{d\tau}{dt} = \frac{1}{\lambda}. \quad (6)$$

Using expressions (4), (5), and (6), we compute the quantity $D(F)$:

$$D(F) = \frac{1}{\lambda} \frac{\partial F}{\partial \tau} + v \frac{\partial F}{\partial r} + \sum_{k=0}^{\infty} \lambda^k \left(A_k \frac{\partial F}{\partial \rho} + B_k \frac{\partial F}{\partial U} + C_k \frac{\partial F}{\partial P} \right). \quad (7)$$

With the aid of (2), we also represent the last expression in the form

$$D(F) = \frac{1}{\lambda^2} \sum_{m=0}^{\infty} \lambda^m \frac{\partial F^{(m)}}{\partial \tau} + \frac{1}{\lambda^2} \sum_{m=1}^{\infty} \lambda^m D^{(m)}(F), \quad (8)$$

where

$$D^{(m)}(F) = v \frac{\partial F^{(m-1)}}{\partial r} + \sum_{k=0}^{m-1} \left(A_k \frac{\partial}{\partial \rho} + B_k \frac{\partial}{\partial U} + C_k \frac{\partial}{\partial P} \right) F^{(m-1-k)}. \quad (9)$$

The right-hand side of the Boltzmann equations can be represented in the form

$$J(F; F) = \frac{1}{\lambda^2} \sum_{m=0}^{\infty} \lambda^m \sum_{k=0}^m J(F^{(k)}; F^{(m-k)}). \quad (10)$$

From expressions (1), (8), and (10) we obtain the following recurrent system of integro-differential equations:

$$\begin{aligned} J(F^{(0)}; F^{(0)}) &= \frac{\partial F^{(0)}}{\partial \tau}, \\ 2J(F^{(0)}; F^{(1)}) &= \frac{\partial F^{(1)}}{\partial \tau} + D^{(1)}(F), \end{aligned} \quad (11)$$

.....

$$\sum_{k=0}^m J(F^{(k)}; F^{(m-k)}) = \frac{\partial F^{(m)}}{\partial \tau} + D^{(m)}(F).$$

The first of the equations of this system can be written in the following simplest form:

$$\frac{\partial F^{(0)}}{\partial \tau} = \frac{s^2}{2m} \iint |(Ve)| \{F'^{(0)} F_1^{(0)} - F^{(0)} F_1^{(0)}\} d\omega dv_1. \quad (12)$$

In the zeroth approximation the distribution function satisfies the spatially homogeneous Boltzmann equation. This equation turned out to be-

the subject of investigations carried out by Carleman ⁽⁸⁾. Equation (12) has a unique solution satisfying the initial data and tending, for sufficiently large τ (as $\tau \rightarrow \infty$), to the Maxwellian distribution function. The process by which the initial distribution function approaches, in the present case, the local Maxwellian distribution function will usually occur very rapidly, after a comparatively small number of collisions. Therefore this initial rapidly proceeding process may be neglected, and the exact system of equations (11) may be replaced by the following approximate system:

$$J(F^{(0)}; F^{(0)}) = 0,$$

$$2J(F^{(0)}; F^{(1)}) = D^{(1)}(F), \tag{13}$$

.....

$$\sum_{k=0}^m J(F^{(k)}; F^{(m-k)}) = D^{(m)}(F).$$

The transition from system (11) to system (13) is equivalent to abandoning the study of phenomena in which the initial data play a significant role.

From the first equation of this system it follows that the distribution function $F^{(0)}$ is a locally Maxwellian distribution function. To determine $F^{(k)}$ from the system of equations (13), the following solvability conditions must be satisfied:

$$\iiint \psi_i D^{(1)}(F) dv = 0, \quad \iiint \psi_i D^{(2)}(F) dv = 0, \dots, \quad \iiint \psi_i D^{(m)}(F) dv = 0. \tag{14}$$

The first of these conditions will be satisfied if $F^{(0)}(r, v, t) = F^0(\rho_0, U^0, P_0)$, where ρ_0, U^0, P_0 satisfy the Euler equations ⁽³⁾.

Let us now consider any one of the following solvability conditions, for example the s -th. From expressions (14) and (9) we shall have

$$\begin{aligned} & \iiint \psi_i D^{(s)}(F) dv = \\ & = \sum_{k=0}^{s-1} \left(A_k \frac{\partial}{\partial \rho} + B_k \frac{\partial}{\partial U} + C_k \frac{\partial}{\partial P} \right) \int_{-\infty}^{+\infty} \int \psi_i F^{(s-1-k)} dv + \frac{\partial}{\partial r_\alpha} \int_{-\infty}^{+\infty} \int \psi_i v_\alpha F^{(s-1)} dv = 0. \end{aligned}$$

On the basis of expression (3), this condition may be written in the following final form:

$$\left(A_{s-1} \frac{\partial}{\partial \rho} + B_{s-1} \frac{\partial}{\partial U} + C_{s-1} \frac{\partial}{\partial P} \right) \int_{-\infty}^{+\infty} \int \int \psi_i F^{(0)} dv + \frac{\partial}{\partial r_\alpha} \int_{-\infty}^{+\infty} \int \int \psi_i v_\alpha F^{(s-1)} dv = 0, \quad (15)$$

where $s = 1, 2, 3, \dots$; $i = 1, 2, 3, 4, 5$.

We shall assume that the solvability conditions are exactly satisfied in all approximations and for all possible values of ψ_i , and we shall use (15) to determine the unknown functions A_k, B_k , and C_k . Setting $\psi_i = 1$, from expression (15) we find $A_{s-1} + \delta_{s1} \partial \rho U_\alpha / \partial r_\alpha = 0$. From this expression it follows that

$$A_0 = -\partial \rho U_\alpha / \partial r_\alpha, \quad A_1 = A_2 = \dots = A_n = 0. \quad (16)$$

Setting $\psi_i = v_i$, from expression (15) we find

$$A_{s-1} U_1 + B_{s-1}^i \rho + \partial P_{i\alpha}^{(s-1)} / \partial r_\alpha = 0, \quad (17)$$

where

$$P_{i\alpha}^{(s-1)} = \int_{-\infty}^{+\infty} \int \int v_i v_\alpha F^{(s-1)} dv.$$

From expressions (16) and (17) we shall have

$$B_0^i = -U_\alpha \frac{\partial U_i}{\partial r_\alpha} - \frac{1}{\rho} \frac{\partial P}{\partial r_i}, \quad B_{s-1}^i = -\frac{1}{\rho} \frac{\partial P_{i\alpha}^{(s-1)}}{\partial r_\alpha}. \quad (18)$$

Putting $\psi_i = v^2$, from (15) we obtain one more solvability condition:

$$\left(A_{s-1} \frac{\partial}{\partial \rho} + B_{s-1} \frac{\partial}{\partial U} + C_{s-1} \frac{\partial}{\partial P} \right) (\rho U^2 + 3P) + \frac{\partial q_\alpha^{(s-1)}}{\partial r_\alpha} = 0. \quad (19)$$

From expressions (19), (18), and (16) we find

$$C_0 = -U_\alpha \frac{\partial P}{\partial r_\alpha} - \frac{5}{3} P \frac{\partial U_\alpha}{\partial r_\alpha}, \quad C_{s-1} = -\frac{1}{3} \frac{\partial q_\alpha^{(s-1)}}{\partial r_\alpha} + \frac{2}{3} U_i \frac{\partial P_{i\alpha}^{(s-1)}}{\partial r_\alpha}, \quad (20)$$

where

$$q_\alpha^{(s-1)} = \int_{-\infty}^{+\infty} \iint v^2 v_\alpha F^{(s-1)} dv.$$

As is seen from the solvability conditions for the inhomogeneous integral equations (15), the aggregate of all unknown functions A_n , B_n , and C_n is completely determined.

We note that in the s -th approximation $s - 1$ solvability conditions must be satisfied and the unknown coefficients up to A_{s-1} , B_{s-1} , C_{s-1} , inclusive, must be determined. In accordance with this, for each approximation there may be determined a system of partial differential equations for the unknown functions $\rho(r, t)$, $U(r, t)$, $P(r, t)$. In the s -th approximation, from expressions (5), (16), (18), and (20), after certain transformations we obtain

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho U_\alpha}{\partial r_\alpha},$$

$$\frac{\partial U_i}{\partial t} = -U_\alpha \frac{\partial U_i}{\partial r_\alpha} - \frac{1}{\rho} \frac{\partial P}{\partial r_i} - \frac{1}{\rho} \sum_{k=0}^{s-1} \lambda^k \frac{\partial P_{i\alpha}^{(k)}}{\partial r_\alpha}, \quad (21)$$

$$\frac{\partial^{3/2} P}{\partial t} = -\frac{\partial^{3/2} P U_\alpha}{\partial r_\alpha} - P \operatorname{div} \sum_{k=1}^{s-1} \lambda^k \left\{ P_{i\alpha}^{(k)} \frac{\partial U_i}{\partial r_\alpha} + \frac{1}{2} \frac{\partial q_\alpha^{(k)}}{\partial r_\alpha} \right\}.$$

Consequently, the aggregate of the functions chosen above $\rho(r, t)$, $U(r, t)$, $P(r, t)$ must satisfy certain equations of hydrodynamics and is determined by its initial data. As is seen from expressions (21), in the first approximation ($s = 1$) we shall have the system of Euler equations for an ideal gas; in the second approximation ($s = 2$), the system of Navier–Stokes equations for a viscous heat-conducting gas; in the third approximation ($s = 3$), a system of equations of the type of Burnett's equations, and so on. If the flow under consideration differs little from the flow of an ideal gas, then $P = P_0 + \lambda P_1 + \dots$, $v = v^{(0)} + \lambda v^{(1)} + \dots$, $T = T_0 + \lambda T_1 + \dots$. In this case the distribution function of normal structure is reduced to the Hilbert distribution function⁽³⁾ by expansion in a series in λ . In its computational technique of successive approximations, this method coincides with Enskog's method. However, in the present method the successive approximations are formed by themselves, whereas in Enskog's method it is necessary artificially to single out a certain aggregate of terms of the series for each approximation.

The modified small-parameter method proposed in the present work, based on Poincaré's ideas, has no restrictions and can be used to solve the Boltzmann equation.

Received
16 IV 1964

CITED LITERATURE

¹ D. Hilbert, *Math. Ann.*, **72**, 562 (1912). ² D. Enskog, Diss., Upsala, 1917. ³ V. V. Struminskii, DAN, **158**, No. 1 (1964). ⁴ H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Paris, 1892. ⁵ M. Lighthill, *Phil. Mag.* (7), **40** (1949). ⁶ Tsien Hsue-shen, *Problems of Mechanics*, Moscow, 1959. ⁷ V. V. Struminskii, DAN, **151**, No. 5 (1963); **153**, No. 3 (1963). ⁸ T. Carleman, *Mathematical Problems of the Kinetic Theory of Gases*, Moscow, 1960.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.