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Abstract

Full Text

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GENERALIZED NORMAL WAVES IN CHAIN STRUCTURES

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I. It is known ([1,2]) that in chains of identical cells that are linear transducers of forced oscillations (transmission lines, cables, cylindrical waveguides, etc.), normal waves are possible, characterized by a constant phase velocity and by invariance of the form of the distribution of oscillation amplitudes in the cells, i.e., along the wave front. Here chains of nonidentical cells are considered. In them, in the general case, normal waves do not exist. However, under certain conditions, forced harmonic oscillations are possible that resemble normal waves in that they partially or completely preserve the forms of oscillation of the cells during propagation along the chains. Special cases of such processes, called generalized normal waves, have been encountered for a long time (see, for example, ([3])).

In the present work: 1) an exact definition and classification of generalized normal waves are given, 2) the conditions and their existence in inhomogeneous chains are clarified, and 3) the possibility is established of expanding arbitrary forced oscillations in terms of them.

Consider a chain of $2N$-terminal networks. Each cell with number j transforms N oscillations $f_{i,\text{in}}(j)e^{i\omega t}$, specified at the input terminals, $i = 1, 2, \dots, N$, into N oscillations $f_{i,\text{out}}(j)e^{i\omega t}$, taken from the output terminals, $i = 1, 2, \dots, N$, and supplied to the input terminals of cell number $j+1$. The set $\{f_i(j)\}_i$ is regarded as the components of a vector $\mathbf{f}(j)$ in the N-dimensional linear space $R^{(N)}$. Specifying the transformation of cell number j in the form $\mathbf{f}_{\text{out}}(j) = \mathbf{A}(j)\mathbf{f}_{\text{in}}(j)$, where $\mathbf{A}(j)$ is a linear operator in $R^{(N)}$, represented by the square matrix $A(j) = \|a'_{ik} + ia''_{ik}\|$, we write the equations for the amplitudes of the oscillations:

$$\mathbf{A}(j)\mathbf{f}(j) - \mathbf{f}(j+1) = \mathbf{P}(j), \qquad j = 0, 1, 2, \dots, \tag{1}$$

where $\mathbf{f}(j) = \mathbf{f}_{\text{in}}(j) = \mathbf{f}_{\text{out}}(j-1)$, and $\mathbf{P}(j)e^{i\omega t}$ are the external actions on the cells.

In a chain free of external actions, the oscillation $\mathbf{f}(0)e^{i\omega t}$ will give rise to a sequence of forced oscillations $\mathbf{f}(j)e^{i\omega t}$, $j = 1, 2, 3, \dots$,

$$\mathbf{f}(j)e^{i\omega t} = \mathbf{A}(j-1)\mathbf{A}(j-2)\cdots\mathbf{A}(1)\mathbf{A}(0)\mathbf{f}(0)e^{i\omega t}, \tag{2}$$

transmitted from cell to cell in such a way that cell number j serves as the source of forced oscillations for the adjacent cell.

II. **Definition 1.** A **generalized normal wave of rank** m (abbreviated, a wave of rank m) is a sequence of oscillations (2), all vectors of which $\{f(j)\}_j = \{f_m(j)\}_j$ lie in a subspace $R^{(m)} \subseteq R^{(N)}$, independent of j, of dimension m, and cannot be represented in the form of the sum

$$\{f_m(j)\}_j = \{\alpha f_{m_1}(j)+\beta f_{m_2}(j)\}_j, \quad \text{where } \{f_{m_1}(j)\} \in R^{(m_1)} \subset R^{(m)},\ \{f_{m_2}(j)\} \in R^{(m_2)} \subset R^{(m)},$$

in particular, cannot be embedded in $R^{(m_1)}$ ($\beta = 0$) or $R^{(m_2)}$ ($\alpha = 0$), $m_1, m_2 < m$.

The normal wave $\mathbf{f}(j)e^{i\omega t} = \mathbf{F}e^{i(\omega t-\gamma j)}$ is a special case of a wave of rank 1.

Definition 2. Waves $\{\mathbf{f}_{m_p}(j)\}_j$, $p = 1, 2, \dots, n$, are called **linearly independent** if

$$\sum_p a_p \mathbf{f}_{m_p}(j) = 0, \quad j = 0, 1, 2, \dots,$$

is possible only when $a_1 = a_2 = \cdots = a_n = 0$.

Lemma 1. The set D of all $\mathbf{f}(0)$ that generate, in $R^{(m)}$, waves of rank m, forms a subspace $R^{(n)} \subseteq R^{(m)}$, where n is the number of linearly independent waves of rank m.

Theorem 1. In order that there exist m linearly independent waves of rank m in $R^{(m)}$, it is necessary and sufficient that the subspace $R^{(m)}$ make the set $\{A(j)\}_j$ reducible and contain no reducing subspaces $R^{(m_1)} \subset R^{(m)}$, except 0. This means that there must exist a transformation C of the vector basis in $R^{(N)}$, independent of j, which brings all $A(j)$ from $\{A(j)\}_j$ to the form

$$A'(j) = C^{-1}A(j)C = \begin{pmatrix} A_m(j) & B \\ 0 & A_{N-m}(j) \end{pmatrix}, \quad j = 0, 1, 2, \dots, \tag{3}$$

where $\{A_m(j)\}_j$ is an irreducible set of matrices of order m.

Definition 3. If in $R^{(m)}$ there is a common invariant subspace $R^{(m_1)}$ for all A_m, and $\{A_m(j)\}_j$ are brought to the form (3), where $m = m_1$ and $N = m$, then waves of rank m_1 in $R^{(m_1)}$ will be called **embedded** in waves of rank m, if R is not equal to 0.

Definition 4. If in $R^{(m)}$ there are two nonintersecting common invariant subspaces $R^{(m_1)}$ and $R^{(m-m_1)}$, i.e. B in Definition 3 is equal to 0 and the set of matrices $\{A_m(j)\}_j$ is decomposable (completely reducible), then the groups of waves of rank $\leq m_1$ in $R^{(m_1)}$ and of rank $\leq m_2$ in $R^{(m_2)}$ will be called **separated**.

If, as a measure of the distortion of the wave form, one takes the greatest distance of the vector $\mathbf{f}_m(j)$ from $\{\mathbf{f}_m(j)\}_j$ to the vector $\mathbf{f}(0)$, then in a sequence of embedded waves of ranks $m_1 < m_2 < \cdots < m$ the distortion of the wave form increases with its rank.

III. Let us formulate, in decreasing order of the rank m, criteria of reducibility, using the terminology introduced above.

1. According to Schur' s lemma (^{4}), in (1) there exist only waves of rank N if the only matrix L commuting with each $A(j)$ from $\{A(j)\}_j$ is the scalar matrix αE.
2. According to Burnside' s theorem (^{4}), if the number of independent matrices in the set formed from $\{A(j)\}_j$ and from the products $A(j_1)A(j_2)\ldots$ is less than N^2, then waves of rank $m < N$ exist in the chain.
3. If the minimal rank of the operator L in criterion 1 is equal to m, then there exist m waves of rank m in the chain.
4. If all $A(j)$ from $\{A(j)\}_j$ commute pairwise, then there exists in the chain at least one wave of rank 1.
5. According to Lie' s theorem (^{5}), if the Lie commutator algebra constructed on the set $\{A(j)\}$ is solvable, then all $A(j)$ are brought to triangular form; this means the existence of one wave of rank 1, one wave of rank not higher than 2, and so on, which may turn out to be embedded in one another.
6. If all $A(j)$ from $\{A(j)\}$ have a simple structure, then criterion 4 is sufficient and necessary for the existence of N waves of rank 1.
7. In a chain of identical cells, when all $A(j)$ are equal and are brought to one and the same Jordan normal form (^{6}), to each Jordan box of order n there correspond one normal wave and $n-1$ consecutively embedded generalized normal waves of ranks $2, 3, \ldots, n$, traveling with identical phase velocities and differing in the increase of amplitudes, occurring according to power laws (spatial resonance (^{2,7})).

IV. **Theorem 2.** *Any forced oscillation in a chain of $2N$-terminal networks, caused by an arbitrary external action with amplitudes $P(j)$ bounded in norm, can be expanded into a spectrum of separated groups of generalized normal waves, where each group consists of a linear combination of embedded waves of the corresponding ranks.*

Proof. Supplement (1) with boundary conditions or with boundedness requirements $\{\|\mathbf{f}(j)\|\}_j$ as $j \to \pm\infty$, in order to ensure a single-valued dependence of the solutions $\mathbf{f}(j)$ on $\{\mathbf{P}(j)\}_j$. Reduce $\{A(j)\}_j$ by one and the same transformation

C to the quasidiagonal form

$$A'(j) = C^{-1}A(j)C = (A_{\alpha\beta}(j)\delta_{\alpha\beta}),$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol, $A_{\alpha\alpha} = A_{m_p}$ are square matrices of order m_p, where $p = 1, 2, \ldots, n$; m_p do not depend on j. Let $\{A_{m_p}(j)\}_j$ be indecomposable (but they may be reducible). In this case the structure of all $A(j)$ is determined up to the sequence of blocks $A_{m_p}(j)$ and up to the choice of bases $\{\mathbf{e}_{i,p}\}_{i=1}^{m_p}$ in the subspaces $R^{(m_p)}$ defined by A_{m_p}. $R^{(N)}$ decomposes into the direct sum $\{R^{(m_p)}\}_p$, i.e.

$$\mathbf{f} = \sum{}' \mathbf{f}^{(p)} \quad \text{and} \quad \mathbf{P} = \sum{}' \mathbf{P}^{(p)},$$

where $\mathbf{f}^{(p)}, \mathbf{P}^{(p)} \in R^{(m_p)}$. For $\mathbf{f}^{(p)}$ we obtain

$$A_{m_p}(j)\mathbf{f}^{(p)}(j) - \mathbf{f}^{(p)}(j+1) = \mathbf{P}^{(p)}(j), \qquad j = 0, 1, 2, \ldots \tag{4}$$

Equation (4) for each p describes a separated group of waves, consisting of m_p linearly independent embedded waves of rank equal to or less than m_p. We represent $\mathbf{P}^{(p)}$ in source form by cells and terminal networks:

$$\mathbf{P}^{(p)}(j) = \sum_{j'}\sum_{i'} P_{i'}^{(p)}(j')\delta_{j,j'}\mathbf{e}_{i',p}. \tag{5}$$

Each unit vector $\delta_{j,j'}\mathbf{e}_{i',p}$, substituted into the right-hand side of (5), generates a particular solution of (4) in the form of a Green function composed of m_p independent embedded waves belonging to the separated group of waves propagating in $R^{(m_p)}$:

$$\mathbf{G}'_p(j, j', i') = \{G'_p(j, j', i, i')\mathbf{e}_{i,p}\}_{i=1}^{m_p} = \sum_{s=1}^{m_p} C'_s(j', i')\mathbf{f}_s^{(p)}(j), \qquad j > j'. \tag{6}$$

The expression for $j < j'$ is analogous. C_s are determined from the boundary conditions (decomposable with respect to $\{R^{(m_p)}\}_p$) and from the matching conditions (6) at $j = j'$. As a result of summing the effects of the unit vectors, we obtain a spectral-source representation of the solution (1) in the form:

$$\mathbf{f}(j)e^{i\omega t} = \left\{\sum_{p=1}^{n}\sum_{j'<j}\sum_{i,i'}^{m_p} P_{i'}^{(p)}(j')\,\mathbf{G}'_p\mathbf{e}_{i,p} + \sum_{p=1}^{n}\sum_{j'>j}\sum_{i,i'}^{m_p} P_{i'}^{(p)}(j')\,\mathbf{G}''_p\mathbf{e}_{i,p}\right\} e^{i\omega t}. \tag{7}$$

For homogeneous chains with matrices of simple structure, the sums over i and i' disappear and (7) passes into the well-known expansion in normal waves $^{(2)}$. If the distance between cells $d \to 0$, then, replacing j by the coordinate $z = \lim_{d\to 0} dj$, and the sums over j' by integrals over z, we pass to continuous chains (N-terminal inhomogeneous lines), for which everything stated above is valid. Definitions 1-4 and Theorems 1 and 2 are valid for infinite-terminal lines, i.e. for inhomogeneous waveguides and layered media, for example spherically layered ones, in which the expansion (7) is carried out in waves of rank 2 (8), separately preserving the forms of the distribution of the fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ along the axis of propagation θ. We note that systems in which waves of rank 1 exist are called in radio engineering *matched with respect to wave impedances*. Chains consisting of homogeneous sections $k = 1, 2, 3, 4, \ldots$, each of which contains N_k identical cells with A_k of simple structure, have become widespread. Generalized normal waves

$(r_1 = 1, 2, \ldots, N)$ in such chains, on each segment k, can be represented in the form of a sum of the normal waves of this segment:

$$Ce^{-i\gamma_{r_1}N_1}\left[\sum_{r_2=1}^{N} R_{r_1 r_2} e^{-i\gamma_{r_2}(N_2-1)} \ldots \left[\sum_{r_k=1}^{N} R_{r_{k-1},r_k}\mathbf{F}^{(r_k)} e^{i(\omega t-\gamma_{r_k} j)}\right] \ldots\right], \qquad (8)$$

where $\gamma_{r_k} = -i\ln\lambda_{r_k}$, λ_{r_k} are the eigenvalues, $\mathbf{F}^{(r_k)}$ are the eigenvectors $\mathbf{A}_k$; $\|R_{r_{k-1}r_k}\|$ $(r_{k-1}, r_k = 1, 2, \ldots, N)$ is the transformation (scattering) matrix of the normal waves of the $(k-1)$-st segment into the normal waves of the k-th segment at the junction of the segments. It is made up of the coefficients of the expansion of $\mathbf{A}_k\mathbf{F}^{(r_{k-1})}$ in $\{\mathbf{F}^{(r_k)}\}_{r_k=1}^{N}$. Representations analogous to (8) are obtained for piecewise homogeneous waveguides and for "piecewise spherical" media (9).

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Note: Figure translations are in progress. See original paper for figures.

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