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Fig. 1

Figure 1: Fig. 1

## Abstract

## Full Text

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*THEORY OF ELASTICITY*

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# DISLOCATION THEOREM

## 1°. General remarks.

In the paper (1) a model of a brittle body was proposed which, in particular, has the following properties: 1) the maximum tensile stresses do not exceed a certain constant of the material $\sigma_0$, called the resistance to separation; 2) the relation between strains and stresses is described by Hooke's law if the tensile stresses do not attain the value $\sigma_0$; 3) cracks form in the model when the maximum stress, determined by the methods of the linear theory of elasticity, exceeds $\sigma_0$. These properties are also preserved in the investigation of crack formation carried out in the present paper.

Here it will be assumed that the crack faces are attracted with a stress $\sigma(2v)$, where $\sigma(2v)$ is an arbitrary continuous monotonically nonincreasing function of the distance between the crack faces $2v$, with $\sigma(0) = \sigma_0$ (Fig. 1). We shall then investigate the brittle fracture of an unbounded body with an inserted or removed material half-plane under uniform tension at infinity by forces perpendicular to the indicated half-plane.

If the body under consideration remained continuous and its deformation everywhere obeyed Hooke' s law, then the normal stresses at points lying on the continuation of the inserted half-plane would be equal to (2)

$$\sigma = S + \frac{E\lambda}{4\pi(1-\nu^2)l}, \tag{1}$$

where $l$ is the distance from the boundary of the inserted half-plane; $\lambda$ is its thickness; $S$ is the tensile stress applied to the body at infinity perpendicular to the inserted half-plane; $E$ is the modulus of elasticity; $\nu$ is Poisson' s ratio.

**Fig. 1**

Fig. 2

Figure 2: Fig. 2

For sufficiently small $l$, the stresses determined by formula (1) will exceed the resistance to separation. Consequently, in this region of the model under consideration a crack forms. We choose the origin of coordinates at the center of the crack, direct the $Ox$ axis along the crack, and the $Oy$ axis perpendicular to it.

The distance between the crack faces is equal to twice the displacement along the $Oy$ axis of the surface points of the crack under the action of the pressure $p(x)$, which supplements the stresses (1) up to the stresses $\sigma(2v)$, i.e.,

$$p(x) = S + \frac{E\lambda}{4\pi(1-\nu^2)(x+L)} - \sigma(2v), \tag{2}$$

where $L$ is the half-length of the crack. The unknowns are the displacement of the crack faces $v$ and the crack length. To determine $v$, following the paper (2), we obtain the relation

$$v(x,+0) = \frac{1-\nu^2}{\pi E}\int_{-L}^{L} p(\xi)\ln\frac{L^2 - x\xi + \sqrt{(L^2-x^2)(L^2-\xi^2)}}{L^2 - x\xi - \sqrt{(L^2-x^2)(L^2-\xi^2)}}\,d\xi. \tag{3}$$

If we substitute here the value $p(x)$, determined by formula (2), and express the function $\sigma(2v)$ explicitly in terms of $2v$, then we obtain an integral equation for finding the unknown displacement $v(x,+0)$.

The length of the slit $2L$ is determined from the equation

$$\left.\frac{\partial v(x,0)}{\partial x}\right|_{x=L} = 0, \tag{4}$$

which follows[3] from the condition that the stresses be bounded ($\sigma \leqslant \sigma_0$).

Equation (3) and condition (4) are sufficient for determining the displacements of the edge of the crack.

**2°. Formulation of the problem.** As a result of the introduction of a material half-plane into the body, a slit is formed. Under the action of tensile stresses $S$, applied to the body at infinity, the crack will develop. For a given fixed thickness $\lambda$ of the introduced layer, it is required to determine the maximum intensity of the load $S$ at which equilibrium of the body under consideration is possible. We shall call the indicated load the limiting load.

**Fig. 2**

Fig. 3

Figure 3: Fig. 3

**3°. Determination of the total and partial surface energy.** Let us imagine an unbounded, ideally homogeneous solid body being split into two half-spaces by normal forces uniformly distributed over the surfaces of the indicated half-spaces. In the process of separation of the solid body it is necessary to overcome the forces of interaction of its particles. These forces are divided into forces of short-range action and forces of Newtonian attraction. The latter are excluded from consideration; therefore, by internal forces we shall mean only forces of short-range action. By the total surface energy $T_0$ of a solid body we shall mean the work expended in overcoming the internal forces in forming a unit of surface. By the partial surface energy $T(\lambda)$ we shall mean the work, referred to a unit of surface, expended in moving the indicated half-spaces away from one another by a distance $\lambda$. The doubled total surface energy is represented by the area (Fig. 1) bounded by the coordinate axes $(\sigma, 2v)$ and the curve $\sigma(2v)$, while the doubled partial surface energy is the area shaded in Fig. 1.

**Fig. 3**

**4°. Dislocation theorem.** *The limiting load is equal to the doubled partial surface energy divided by the thickness of the introduced layer, i.e.*

$$S = \frac{2T(\lambda)}{\lambda}. \tag{5}$$

**5°. Proof of the theorem.** Let us replace the dependence $\sigma(2v)$ (Fig. 1) by a broken line, as shown in Fig. 2. For the time being we shall assume that

the edges of the crack either repel or attract each other with stress $\sigma_0 i/n$ ($i = 1, 2, \dots, n$), if the distance between them lies in the interval from $\eta_{n-i}$ to $\eta_{n-i+1}$, or do not interact in the opposite case; here $n$ is an arbitrary integer, and $\eta_i$ are any quantities (with the dimension of length) satisfying the inequalities

$$\eta_0 = 0 < \eta_1 < \eta_2 < \dots < \eta_{n-1} < \eta_n.$$

For the time being consider the case $\lambda < \eta_n$. In this case the edges of the crack interact as shown in Fig. 3. The boundary conditions for such a crack will be

$$\sigma(2v) = \begin{cases} k\sigma_0/n & \text{for } -L < x < b_{k+1}, \\ (k+1)\sigma_0/n & \text{for } b_{k+1} < x < b_{k+2}, \\ (k+2)\sigma_0/n & \text{for } b_{k+2} < x < b_{k+3}, \\ \dots & \dots \\ \sigma_0 & \text{for } b_n < x < L, \end{cases} \tag{6}$$

where $k$ is an integer ($k \geq n$) depending on the magnitude of $\lambda$.

The unknown quantities $L, b_i$ ($i = k+1, k+2, \dots, n$) are determined from equation (4) and from the following conditions: at the points where the distance $2v$ between the surfaces of the crack is equal to $\eta_{n-i+1}$ ($i = k+1, k+2, \dots, n$), the force of interaction between the edges of the crack changes by the amount $\sigma_0/n$ (Fig. 2). The abscissas of these points are denoted in Fig. 3 by $b_i$; consequently, one must have

$$2v(b_{k+1}, +0) = \eta_{n-k}, \quad 2v(b_{k+2}, +0) = \eta_{n-k-1}, \dots, \quad 2v(b_n, +0) = \eta_1. \tag{7}$$

Equations (4) and (7), using formulas (2), (3), (6) and the results of work ($^4$), give

$$S = \sigma_0 \frac{a\left[\pi k + \sum\limits_{j=k+1}^{n} (\sin\beta_j + \beta_j)\right] + \pi k \sum\limits_{j=k+1}^{n} (\sin\beta_j - \beta_j) - \left(\sum\limits_{j=k+1}^{n} \beta_j\right)^2}{\pi n \left[\sum\limits_{j=k+1}^{n} (\sin\beta_j - \beta_j) + a\right]}, \tag{8}$$

where

$$\beta_i = \arccos\frac{b_i}{L} \quad (i = k+1, k+2, \dots, n), \qquad a = \frac{\pi}{\lambda}\sum_{j=1}^{n-k} \eta_j,$$

with the principal values of $\beta_i$ understood.

The limiting load in the present case ($\lambda < \eta_n$) coincides with the maximum of the right-hand side of formula (8), considered as a function of the variables $\beta_i$. From the indicated formula it follows that

$$\left.\frac{\partial S}{\partial z}\right|_{z=a} = 0, \qquad \left.\frac{\partial^2 S}{\partial z^2}\right|_{z=a} < 0, \tag{9}$$

where

$$z = \sum_{j=k+1}^{n} \beta_j.$$

The relations (9) hold for arbitrary values of $\beta_i$ ($0 < \beta_i < \pi$). Moreover, for $z = a$ the right-hand side of formula (8) does not depend on $\beta_i$. Consequently, it has a maximum at $z = a$. This maximum is represented in the form

$$S = \frac{\sigma_0}{n}\left(\frac{1}{\lambda}\sum_{j=1}^{n-k}\eta_j + k\right). \tag{10}$$

The limiting load for $\lambda > \eta_n$ is determined by formula (10), in which one should set $k = 0$, i.e.

$$S = \frac{\sigma_0}{n\lambda}\sum_{j=1}^{n}\eta_j. \tag{11}$$

The partial and total surface energies under a multistep law of interaction of the crack edges are represented in the form

$$T(\lambda) = \frac{\sigma_0}{2n}\left(\sum_{j=1}^{n-k}\eta_j + \lambda k\right) \quad (\eta_{n-k} < \lambda < \eta_{n-k+1}), \tag{12}$$

$$T_0 = \frac{\sigma_0}{2n}\sum_{j=1}^{n}\eta_j. \tag{13}$$

It is obvious that for $\lambda \geqslant \eta_n$ we have $T(\lambda)\big|_{\lambda\geqslant\eta_n} = T_0$.

Comparing equalities (10) and (11) with relations (12) and (13), we arrive at formula (5). Thus, the dislocation theorem has been proved for a multistep law of interaction of the crack edges for arbitrary values of $n$ and $\eta_i$. By specifying the corresponding value of $n$ and the law of variation of $\eta_i$ (as a function of $i$), one can obtain from the multistep law of attraction of the crack edges any prescribed dependence of the interaction of the crack surfaces, which proves the theorem in the general case.

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*Note: Figure translations are in progress. See original paper for figures.*

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