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Abstract

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A METHOD OF SUCCESSIVE APPROXIMATIONS FOR FINDING THE POLYNOMIAL OF BEST APPROXIMATION

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1. Let X be a normed space (over the field of all real or all complex numbers); let $x_1, x_2, \dots, x_n$ be linearly independent vectors in X; denote by $\Omega = \mathcal{L}(\{x_k\})$ the elements of Ω—linear combinations of the vectors $x_1, x_2, \dots, x_n$ —which we shall call **polynomials**; if u is a given element of the space X, then a polynomial Q such that, for $P = Q$, the functional $\|u-P\|$ $(P \in \Omega)$ attains its least value $\mu = \rho(u, \Omega)$, is called the **polynomial of best approximation** (to u), and the number μ the **best approximation**.

2. The process of determining the polynomial of best approximation consists in the successive solution of a number of auxiliary problems: let $g_k \in X^*$ $(k = 1, 2, \dots, m;\ m \geq n)$; it is required to determine $P' \in \Omega$ such that

$$\max_{k \leq m} |g_k(P' - u)| = \min_{P \in \Omega} \left[\max_{k \leq m} |g_k(P - u)| \right]. \qquad (1)$$

As is known, such problems can be solved by means of linear programming, for example with the aid of a certain modification of the simplex method.

3. The construction of the polynomial of best approximation Q and the determination of the best approximation μ are carried out according to the following scheme.

Denote by X_0 the subspace of the space X spanned by the set Ω and the element u, and let $\Gamma \subset X^*$ be a set of normalized functionals having the property that for every $x \in X_0$ one can specify such an $f_0 \in \Gamma$ that $|f_0(x)| = \|x\|$.

As is easy to see, in Γ there exist elements $f_1, f_2, \dots, f_n$ such that the determinant $|f_i(x_k)|_{i,k=1,2,\dots,n} \neq 0$, and then, evidently, there exists an "interpolation" polynomial $P_n \in \Omega$, determined by the equalities

$$f_i(u - P_n) = 0 \qquad (i = 1, 2, \dots, n).$$

It is clear that P_n is a solution of problem (1), if in it one takes $m = n,\ g_k = f_k$ $(k = 1, 2, \dots, n)$.

Let, for $m \geq n$, functionals $f_1, f_2, \dots, f_m \in \Gamma$ and polynomials $P_n, P_{n+1}, \dots, P_m \in \Omega$ be defined so that

$$\max_{k \leq s} |f_k(P_s - u)| = \min_{P \in \Omega} \left[\max_{k \leq s} |f_k(P - u)|\right] \qquad (s = n, n+1, \dots, m), \qquad (2)$$

$$|f_{k+1}(P_k - u)| = \|P_k - u\| \qquad (k = n, n+1, \dots, m-1). \qquad (3)$$

The functional f_{m+1} is found so that (3) is satisfied for $k = m$, and the polynomial P_{m+1} is determined as the solution of problem (1), where $g_k = f_k$ $(k = 1, 2, \dots, m+1)$.

As a result of the process described, two sequences are obtained: $\{f_k\}$ $(f_k \in \Gamma;\ k = 1, 2, \dots)$ and $\{P_s\}$ $(P_s \in \Omega;\ s = n, n+1, \dots)$ such that (2) and (3) are satisfied for all $k,\ s \geq n$. Denote $\mu_m = \|P_m - u\|$ $(m \geq n)$.

Theorem. The sequence $\{\mu_m\}$ converges to μ. The sequence $\{P_m\}$ is bounded (in the space X). If $\{P_{m_i}\}$ is any convergent subsequence of the sequence $\{P_m\}$, then $Q = \lim_{i \to \infty} P_{m_i}$ is a polynomial of best approximation. If the polynomial of best approximation Q is unique, then the sequence $\{P_m\}$ itself converges to Q.

4. Before proving the theorem we give some preliminary considerations.

In the space X introduce seminorms $\|\cdot\|_s$ $(s \geq n)$, putting

$$\|x\|_s = \max_{k \leq s} |f_k(x)|.$$

Clearly,

$$\|x\|_s \leq \|x\|_{s+1} \leq \|x\| \qquad (x \in X;\ s = n, n+1, \dots). \qquad (4)$$

Lemma 1. If $s > m$, then $\mu_m = \|P_m^* - u\|_s$.

Next, denote

$$\lambda_m = \|P_m - u\|_m \qquad (m \geq n).$$

Lemma 2. $\lambda_n \leq \lambda_{n+1} \leq \cdots$.

Lemma 3. $\lambda_m \leq \mu \leq \mu_m$ $(m \geq n)$.

Remark. If it turns out that $\lambda_m = \mu_m$ for some m, then $\mu = \lambda_m = \mu_m$ and P_m will be a polynomial of best approximation. In this case the polynomials P_k $(k \geq m)$ will also be polynomials of best approximation.

5. **Proof of the theorem.** It is not hard to see that in the space X_0 the seminorm $\|\cdot\|_{n+1}$ will be a norm. But then, since X_0 is finite-dimensional, there is a number K such that $\|x\| \leq K\|x\|_{n+1}$ $(x \in X_0)$; then, on the basis of Lemma 3 and (4),

$$\mu_m = \|P_m - u\| \leq K\|P_m^* - u\|_{n+1} \leq K\|P_m - u\|_m = K\lambda_m \leq K\mu \qquad (m \geq n+1),$$

so that the sequence $\{\mu_m\}$, and hence also the sequence $\{P_m\}$, is bounded. Consider an arbitrary subsequence $\{\mu_{m_i}\}$ and the corresponding sequence $\{P_{m_i}\}$. Passing, if necessary, to a subsequence of this sequence, we may assume that there exists

$$Q = \lim_{i\to\infty} P_{m_i}.$$

Then

$$\mu_{m_i} = \|P_{m_i} - u\| \to \|Q - u\|.$$

Further,

$$\lambda_{m_i} \to \lambda = \lim_{m\to\infty} \lambda_m.$$

According to Lemma 1, taking (4) into account, we may write

$$\begin{aligned} \mu_{m_i} &= \|P_{m_i} - u\|_{m_i+1} \leq \|P_{m_{i+1}} - u\|_{m_i+1} + \|P_{m_{i+1}} - P_{m_i}\|_{m_i+1} \\ &\leq \lambda_{m_{i+1}} + \|P_{m_{i+1}} - P_{m_i}\| \qquad (i = 1, 2, \ldots). \end{aligned}$$

Passing here to the limit, we obtain

$$\mu \leq \|Q - u\| \leq \lambda.$$

But, by Lemma 3, $\lambda \leq \mu$. Thus $\|Q - u\| = \mu$, $\mu_{m_i} \to \mu$. It follows that $\mu_m \to \mu$.

The remaining assertions of the theorem follow in an obvious way from what was said above.

6. **Remarks.**

1°. Since $\lambda_m \to \lambda = \mu$, the result of Lemma 3 can be used to estimate the rate of convergence $\mu_m \to \mu$ (and, of course, $\lambda_m \to \mu$).

2°. The method described above for finding a polynomial of best approximation can also be used in the case when it is required that the coefficients $a_1, a_2, \ldots, a_n$ of the desired polynomial satisfy certain linear restrictions:

$$\sum_{k=1}^{n} c_i^k a_k \leq C_i \qquad (i = 1, 2, \ldots, \nu). \tag{5}$$

In this case it is only necessary to replace the set Ω by the set of all such polynomials whose coefficients satisfy conditions analogous to (5).

3°. If the process described above is modified somewhat, then in the role of Γ one may consider such a set of normalized functionals that, for each $x \in X_0$, one has

$$\|x\| = \sup_{f \in \Gamma} |f(x)|.$$

4°. Let us indicate some particular cases.

a) $X = C(T)$, where T is a compact topological space. As Γ here one may take the set T, i.e., more precisely, the set of all positive measures with singleton supports.

b) If T is a closed bounded set in Euclidean space and $X = C^{(r)}(T)$, then one may take $\Gamma = T \times \{0, 1, \dots, r\}$.

c) If $X = L^p$ $(1 \leq p \leq \infty)$, then in the case $p < \infty$ one should take for Γ the set of all normalized functionals. For $p = \infty$, it is expedient to understand Γ as the set of all absolutely continuous positive normalized measures.

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Note: Figure translations are in progress. See original paper for figures.