LAX THEOREMS FOR NONLINEAR EVOLUTION EQUATIONS

L. I. Yakut

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Abstract

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In the note (1), Lax' s theorem (see (2)) on the convergence of stable difference schemes was refined and generalized to certain classes of evolution equations. However, in application to nonlinear equations, the results of (1) allowed one only to consider nonlinearities of a special structure, and the stability condition was formulated in a form that could be verified for partial differential equations only in the presence of a maximum principle. In the present article broader classes of nonlinear equations are studied: equations with a linear principal part

$$du/dt + A(t)u = \varphi(t,u) \tag{1}$$

($u(t)$ is the unknown function) and the quasilinear equation

$$du/dt + B(t,u)u = 0. \tag{2}$$

Let us first consider an equation of the form

$$du/dt + A(t)u = f(t) \qquad (0 \le t \le T), \tag{3}$$

where $A(t)$, for each $t \in [0,T]$, is a linear unbounded closed operator acting in a Banach space E, with dense domain of definition $D(A)$, independent of t; $f(t)$ is a given function satisfying the initial condition

$$u(0) = u_0. \tag{4}$$

Suppose that the space E contains narrower Banach spaces E_1 and E_2 such that $E_2 \subset E_1 \subset E$ and $D(A^\alpha(t)) \subset E_2$ for some $\alpha > 0$, and moreover

$$\|v\|_{E_2} \le C_1 \|A^\alpha(t)v\|_E \qquad (v \in D[A^\alpha(t)])\,. \tag{5}$$

Let there exist a sequence of bounded operators $A_n(t)$ acting in E_2 such that the following consistency condition holds:

C. For every $v \in E_2$,

$$\sup_{0\le t\le T}\left\|(A_n(t)-A(t))A^{-1}(t)v\right\|_{E_1}\le \rho_n\|v\|_{E_2},$$

where ρ_n does not depend on v and $\rho_n\to 0$ as $n\to\infty$.

Let the subspace L_n of the space E_2, consisting of all elements of E_2 on which the operator $A_n(t)$ vanishes, be independent of t. By S_n denote the quotient space of the space E_2 by the subspace L_n: $S_n=E_2/L_n$. We introduce the norm in the space S_n by the formula

$$\|\bar v\|_{S_n}=\inf_{v\in\bar v}\|v\|_{E_1}.$$

We shall assume that, for elements of the space E_2, one can introduce a degenerate norm (seminorm) $\|\cdot\|_n$ such that:

1°. $\|v\|_n=0$ if and only if $v\in L_n$.

2°. $\|v\|_n\le C_2\|v\|_{E_1}$ (C_2 does not depend on n).

3°. $\|v\|_n\le\|v\|_E+\varepsilon_n\|v\|_{E_2}$ ($\varepsilon_n\to 0$ as $n\to\infty$).

From 1° it follows that in the space S_n the norm may also be introduced in the following way: $\|\bar v\|_n=\|v\|_n$, where $v\in\bar v$. By virtue of property 2°, the norm $\|\cdot\|_{S_n}$ majorizes the norm $\|\cdot\|_n$: $\|\bar v\|_n\le C_2\|\bar v\|_{S_n}$.

Let, in addition, the following hold:

4°. $\|\bar v\|_{S_n}\le\dfrac{1}{\gamma_n}\|\bar v\|_n$, where $\gamma_n\to 0$ as $n\to\infty$.

The operator $A_n(t)$ naturally generates in the space S_n an operator $\overline{A_n(t)}$ by the formula $\overline{A_n(t)}\,\bar v=\overline{A_n(t)v}$.

We shall assume that $f(t)$ is a function with values in E_2 and $u_0\in E_2$; then in the space S_n one can construct a finite-difference analogue of problem (3)–(4):

$$\frac{\bar v_{k+1}-\bar v_k}{\Delta_n^t}+\overline{A_n(k\Delta_n^t)}\,\bar v_k=\bar f_k, \tag{6}$$

$$\bar v_0=\bar u_0. \tag{7}$$

For problem (6)–(7) there are two types of theorems: in theorems of weak type, the fulfillment of the stability condition in the norm $\|\ \|_n$ implies the convergence of approximate solutions to the exact one in the norm $\|\ \|_n$; in theorems of strong type, stability in the weak norm implies convergence in the norm $\|\ \|_{S_n}$.

We shall consider problem (3)–(4) under the assumption that the operator $A(t)$, for any λ with $\operatorname{Re}\lambda\geqslant 0$, has a resolvent $(A(t)+\lambda I)^{-1}$, and

$$\left\|(A(t)+\lambda I)^{-1}\right\|_E \leqslant \frac{C}{|\lambda|+1} \quad (0 \leqslant t \leqslant T)$$

(see (3)) and that condition (5) is fulfilled for $\alpha = \gamma_1 < \gamma_2 \in (0,1)$. Under these assumptions the following theorem of strong type holds.

Theorem 1. *Suppose that the consistency condition C and the stability condition*

$$. \quad \left\|1-\Delta_n^t\,\overline{A_n(t)}\right\|_n \leqslant 1 + C\Delta_n^t \quad \text{for } t \in [0,T].$$

are fulfilled.

Suppose that the function $f(t)$ is uniformly bounded in E_2 and continuously differentiable in E.

Suppose that the operator $A(t)A^{-1}(0)$ is strongly continuously differentiable in E. Then, if

$$u_0 \in D[A^{1+\gamma_2}(0)]\,, \quad f(0) \in D[A^{\gamma_2}(0)]$$

and

$$\rho_n = o(\gamma_n), \quad \Delta_n^{t\,\gamma_2} = o(\gamma_n), \quad \varepsilon_n \Delta_n^{t\,\gamma_2-\gamma_1} = o(\gamma_n),$$

then the solution of problem (6)–(7) converges to the solution of problem (3)–(4) in the sense that, as $\Delta_n^t \to 0$,

$$\left\|\overline{u(t)} - \bar{v}_{k_n}\right\|_{S_n} \to 0$$

uniformly with respect to $t \in [0,T]$.

Let us now turn to problem (1)–(4). Suppose that the operator A in equation (1) satisfies the conditions stated above. For the construction of finite-difference equations, let us assume that the right-hand side of equation (1) has the following property:

Let $\bar{v}$ be some equivalence class from the space S_n: $\bar{v} \in S_n$. All elements $v \in \bar{v}$ with norm $\|v\|_{E_2} \leqslant R_2$ are mapped by the operator $\varphi(t,v)$, for each fixed t, into one and the same class from S_n.

In this case problem (1)–(4) corresponds, in the space S_n, to the finite-difference problem

$$\frac{\bar{v}_{k+1} - \bar{v}_k}{\Delta_n^t} + \bar{A}_n \bar{v}_k = \bar{\varphi}_k, \qquad \bar{v}_0 = \bar{u}(0),$$

where $\bar{\varphi}_k = \overline{\varphi(k\Delta_n^t, \bar{v}_k)}$, and $\bar{v}_k \in \bar{v}_k$.

Theorem 2. *Suppose $u(t)$ is a solution of equation (1) such that*

$$u(0) = u_0 \in D[A^{1+\gamma_2}(0)]\,,$$

and the function $f(t) = \varphi(t, u(t))$ and the operators $A(t)$, $\bar{A}_n(t)$ satisfy all the conditions of Theorem 1. Suppose, in addition, that on the ball of the space E_1 of radius $R > R_u$, where

$$R_u = \max_{0 \leqslant t \leqslant T} \|u(t)\|_{E_1},$$

the function $\varphi(t, v)$ satisfies the condition

$$\|\varphi(t, v) - \varphi(t, w)\|_n \leqslant C_R \|v - w\|_n.$$

If

$$\varepsilon_n \Delta_n^t{}^{\gamma_2-\gamma_1} = o(\gamma_n), \quad \rho_n = o(\gamma_n), \quad \Delta_n^t{}^{\gamma_2} = o(\gamma_n) \quad (\gamma_2 > \gamma_1;\ \gamma_2, \gamma_1 \in (0,1)),$$

then the solution of the finite-difference problem for (1)–(4) converges to the solution

$u(t)$ in the sense that, as $\Delta_n t \to 0$,

$$\left\| u(t) - \bar{v}_{k_n} \right\|_{S_n} \to 0$$

uniformly in $t \in [0, T]$.

Theorem 3. Let $u(t)$ be a solution of equation (1) such that $u(0) = u_0 \in D[A^{\gamma_1}(0)]$, and let the function $f(t) = \varphi(t, u(t))$ satisfy all the conditions of Theorem 1. Suppose that the consistency condition C and the stability condition

$$\text{Y}'. \quad \left\| 1 - \Delta_n t\, \bar{A}_n(t) \right\|_{S_n} \leq 1 + C \Delta_n t \quad \text{for all } t \in [0, T]$$

are fulfilled. Let the operator $A(t)A^{-1}(0)$ be strongly continuously differentiable in E. Suppose, moreover, that on the ball of the space E_1 of radius $R_1 > R_u$ the function $\varphi(t, v)$ satisfies the condition

$$\|\varphi(t, v) - \varphi(t, w)\|_{E_1} \leq C_{R_1} \|v - w\|_{E_1}.$$

Then the solution of the finite-difference problem for (1)–(4) converges to the solution $u(t)$ in the sense that, as $\Delta_n t \to 0$,

$$\left\| u(t) - \bar{v}_{k_n} \right\|_{S_n} \to 0$$

uniformly in $t \in [0, T]$.

We now consider equation (2). Let $u(t)$ be a solution of this equation with values in the ball of radius R_u of the space E_1: $\|u(t)\|_{E_1} \leq R_u$. Assume that the operator $A(t) = B(t, u(t))$ satisfies, for all $t \in [0, T]$, the conditions stated above. We shall suppose that the operator $B(t, w)$ is defined on $D(A(0))$ for any $w \in E_1$. Suppose that in the space E_1 there exists a sequence of continuous bounded operators $B_n(t, w)$, depending on elements $w \in E_1$, such that the following consistency condition holds:

C. For any $v \in E_2$ and w from the ball T_u of radius $R_1 > R_u$ of the space E_1,

$$\sup_{0 \leq t \leq T} \left\| (B_n(t, w) - B(t, w)) A^{-1}(t) v \right\|_{E_1} \leq \rho_n \|v\|_{E_2},$$

where ρ_n does not depend on v and $\rho_n \to 0$ as $n \to \infty$.

Introduce the operator $\bar{B}_n(t, \bar{v})$ by the formula

$$\bar{B}_n(t, \bar{v}) \bar{w} = \overline{B_n(t, v) w},$$

and we shall assume that the operator $\overline{B_n(t, v)}$ is defined in the same way for any $v \in \bar{v}$, and construct in the space S_n the finite-difference problem

$$\frac{\bar{v}_{k+1} - \bar{v}_k}{\Delta_n t} + \overline{B_n(k \Delta_n t, \bar{v}_k)}\, \bar{v}_k = 0 \quad (v \in \bar{v}_k), \tag{8}$$

$$\bar{v}_0 = \bar{u}_0. \tag{9}$$

Theorem 4. Suppose that the consistency condition C and the stability condition

$$\text{Y.} \quad \left\| 1 - \Delta_n t\, \overline{B_n(t, v)} \right\|_n \leq 1 + C \Delta_n t \quad \text{for all } t \in [0, T] \text{ and } v \in T_u$$

are fulfilled. Suppose that in the ball T_u the condition

$$\left\| [B(t, v) - B(t, w)] A^{-1-\gamma_1}(t) z \right\|_n \leq C_{R_1} \|v - w\|_n \|z\|_E \quad (z \in E)$$

is satisfied. Then, if $u_0 \in D[A^{1+\gamma_2}(0)]$ and $\varepsilon_n \Delta_n t^{\gamma_2 - \gamma_1} = o(\gamma_n)$, $\rho_n = o(\gamma_n)$, $\Delta_n t^{\gamma_2} = o(\gamma_n)$, the solution of the finite-difference problem (8)–(9) converges to the solution $u(t)$ in the sense that, as $\Delta_n t \to 0$,

$$\left\| \overline{u(t)} - \bar{v}_{k_n} \right\|_{S_n} \to 0$$

uniformly in $t \in [0, T]$.

The results obtained are applied to the proof of convergence of stable finite-difference schemes for solving problems

$$\partial u/\partial t + \mathcal{L}u = f,$$

where $\mathcal{L}$ is a strongly elliptic operator of order $2m$ with sufficiently smooth coefficients, defined in a bounded domain G of s-dimensional space. The space E is the space $L_p(G)$, $E_1 = C(G)$, $E_2 = C^{l+\alpha}(G)$. Let K_n be some cubulation of s-dimensional space; then

$$\|u\|_{S_n} = \max_{x \in K_n \subset G} |u(x)|, \qquad \|u\|_{h,p} = \left[h^s \sum_{x \in K_n \subset G} |u(x)|^p \right]^{1/p},$$

where h is the edge of an elementary cube of the cubulation K_n.

For the application of the abstract results to concrete equations, one has to use the embedding theorems of S. L. Sobolev (^{4}), a priori estimates for solutions of elliptic equations (^{5}), estimates of the resolvent of elliptic operators (^{6}), and theorems on fractional powers of elliptic operators (^{7}).

We shall state one theorem for the first boundary-value problem for equations of the form

$$\partial u/\partial t + \mathcal{L}(x, t, u)u = 0, \tag{10}$$

where $\mathcal{L}(x, t, u)$ is a quasilinear elliptic operator with coefficients depending only on the unknown function u.

We shall say that the finite-difference operators $\mathcal{L}_n(t, v)$, constructed on the cubulation K_n, satisfy uniformly in $t \in [0, T]$ and v with $|v| < R_1$ the consistency condition of order $l + \alpha$ with the operator $\mathcal{L}$, if for every function satisfying the boundary conditions and having derivatives of order $2m + l$ belonging to the Hölder space $C^\alpha(G)$, one has

$$\|\mathcal{L}_n(t, v)u_0 - \mathcal{L}(t, v)u_0\|_C \leq K h^{l+\alpha} \|u_0\|_{C^{2m+l+\alpha}}.$$

Theorem 5. *Let the coefficients of the expression $\mathcal{L}$ depend sufficiently smoothly on x and satisfy, in the aggregate of the variables $t \in [0, T]$ and v with $|v| < R_1$, a Hölder condition uniformly in x, with exponents γ and 1, respectively. Let the operators $\mathcal{L}_n(t, v)$ satisfy the consistency condition of order $l + \alpha$ and the stability condition*

$$\|(1 - \Delta_n t\, \mathcal{L}_n(t, v))\varphi\|_{n,q} \leq (1 + C\Delta_n t)\|\varphi\|_{n,q}$$

for $q > s/(l+\alpha)$, *for all* $t \in [0,T]$ *and* v *with* $|v| < R_1$. *Let the function* $u_0(x) \in W_p^{4m}(G)$ $(p \geq q)$; *let* $u_0, \mathcal{L}u_0$ *satisfy the boundary conditions.*

If

$$\gamma p > \frac{s+(l+\alpha)p}{2m} \quad \text{and} \quad \Delta_n t = O(h^\beta), \quad \text{where} \;\; \beta > \max\left(\frac{s}{q\gamma}, \frac{s-\alpha}{q(\gamma-\gamma_1)}\right),$$

$$\gamma_1 > \frac{s+(l+\alpha)p}{2mp},$$

then the solutions of the finite-difference problem

$$-\frac{v(t_{r+1},x_s)-v(t_r,x_s)}{\Delta_n t} + \mathcal{L}_n(t_r,x_s,v_{rs})\,v(t_r,x_s) = 0,$$

$$v(0,x_s) = u_0(x_s)$$

converge uniformly in t *and* x *to the solution of equation (10) satisfying the boundary conditions and the initial condition*

$$u(0,x) = u_0(x).$$

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Voronezh State University

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Note: Figure translations are in progress. See original paper for figures.