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Abstract

Full Text

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ON FOURIER SERIES OF CONTINUOUS FUNCTIONS WITH RESPECT TO THE HAAR SYSTEM

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In the paper of P. L. Ul' yanov ([5]) some properties of series with respect to the Haar system $\{\chi_n(t)\}$ are formulated (see ([1]), p. 48), some of which are analogous to, and others essentially different from, the properties of trigonometric series. In the present note we give some results concerning the properties of Fourier series of continuous functions with respect to the system $\{\chi_n(t)\}$.

§ 1. Let $f(t)$ be a continuous function on $[0, 1]$. Put

$$c_n(f) = \int_0^1 f(t)\chi_n(t)\,dt \qquad (n = 1, 2, ...).$$

From the results of Chiselskii and Musielak (see ([7]), theorem 2) it follows that if $\omega(\delta, f)$ is the modulus of continuity of the function $f(t)$, then the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\,\omega\left(\frac{1}{n}, f\right)$$

implies the convergence of the series

$$\sum_{n=1}^{\infty} |c_n(f)|^*.$$

This assertion, generally speaking, cannot be strengthened, as the following theorem shows.

Theorem 1. If a function $\omega(\delta)$ is given and, as $\delta \downarrow 0$, we have $\omega(\delta) \downarrow 0$, $\frac{1}{\delta}\omega(\delta) \uparrow$ **, and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\,\omega\left(\frac{1}{n}\right) = \infty, \tag{1}$$

then there exists a continuous function $f(t)$ for which

$$\text{a)}\ \ \omega(\delta, f) = O(\omega(\delta)); \qquad \text{b)}\ \ \sum_{n=1}^{\infty} |c_n(f)| = \infty. \tag{2}$$

Theorem 1 is analogous to S. B. Stechkin' s theorem for trigonometric series (see ([8]) and ([2]), p. 625). From it, taking the footnote into account, follows

Corollary 1. If $\omega(\delta)$ is a modulus of continuity and (1) is satisfied, then there exists a continuous function $f(t)$ for which (2) is valid.

Let us note that from the result of Chiselskii and Musielak it follows that for functions $f(t) \in \mathrm{Lip}\,\alpha$ with $\alpha > 1/2$ one always has

$$\sum_{n=1}^{\infty} |c_n(f)| < \infty.$$

This result is final, since from Corollary 1 we obtain

Corollary 2*. There exists a function $f(t) \in \mathrm{Lip}\,1/2$ for which

$$\sum_{n=1}^{\infty} |c_n(f)| = \infty.$$

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* For the trigonometric system such a result was proved by S. N. Bernstein (see ([2]), p. 608).

** In fact, it suffices to require only $\omega(\delta)/\delta \leq C\omega(\eta)/\eta$ for $0 < \eta < \delta \leq 1$, where C is a constant.

*** P. L. Ul' yanov ([6]) proved the assertion of Corollary 2 for a bounded function $f(t)$ whose integral modulus of continuity is $\omega_1(\delta, f) = O(\delta^{1/2})$.

This assertion is analogous to S. N. Bernstein' s theorem for the trigonometric system (see ([2]), p. 623).

Put now

$$E_n(f) = \inf_{\{\alpha_k\}} \sup_{0 \leq t \leq 1} \left| f(t) - \sum_{k=1}^{n} \alpha_k \chi_k(t) \right|.$$

Theorem 2. The convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} E_n(f)$$

implies the convergence of the series

$$\sum_{n=1}^{\infty} |c_n(f)|.$$

This theorem, generally speaking, cannot be strengthened, for the following is true.

Theorem 3. If a sequence $E_n \downarrow 0$ is given and

$$\alpha) \quad nE_n \uparrow; \qquad \beta) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} E_n = \infty,$$

then there exists a continuous function $f(t)$ for which

$$\text{a) } E_n(f) = O(E_n); \qquad \text{b) } \sum_{n=1}^{\infty} |c_n(f)| = \infty.$$

Theorems 2 and 3 are similar to S. N. Bernstein's theorems for the trigonometric system (see ([2]), Chap. IX).

For functions of the class $\operatorname{Lip}\alpha$ the following holds.

Theorem 4. In order that a continuous function $f(t) \in \operatorname{Lip}\alpha$ for some $0 < \alpha \leq 1$, it is necessary and sufficient that each of the following conditions hold:

$$1) \quad E_n(f) = O(n^{-\alpha}); \qquad 2) \quad \sup_{0 \leq t \leq 1} \left| f(t) - \sum_{k=1}^{n} c_k(f)\chi_k(t) \right| = O(n^{-\alpha}).$$

For the trigonometric system, such a result with condition 1) for $0 < \alpha < 1$ (the case $\alpha = 1$ has special features) was proved by Jackson and S. N. Bernstein (see ([4]), p. 135).

We note that the necessity of both conditions of the theorem follows from N. B. Nady' s result (see ([1]), p. 259).

§ 2. We formulate one theorem on functions of the class $V_p(0,1)$ for $p \geq 1$. The class $V_p(0,1)$ is the set of all functions $f(t)$ such that

$$\sup_{0=t_0<t_1<\cdots<t_n=1} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \right\}^{1/p} < \infty.$$

Theorem 5. 1) If $f(t) \in V_p(0,1)$ for some $p \geq 1$, then for every $\beta > \dfrac{2p}{2+p}$ and $\gamma < \dfrac{1}{p}$ we have

$$\text{a) } \sum_{n=1}^{\infty} |c_n(f)|^{\beta} < \infty; \qquad \text{b) } \sum_{n=1}^{\infty} n^{\gamma-1/2} |c_n(f)| < \infty. \tag{3}$$

2) For every $p \geq 1$ there exists a function $\varphi(t) = \varphi_p(t) \in V_p(0,1)$ (moreover, $\varphi_p(t) \in \operatorname{Lip} \frac{1}{p}$), for which (3) is not fulfilled when $\beta = \dfrac{2p}{2+p}$ and $\gamma = \dfrac{1}{p}$.

Since $\operatorname{Lip} \alpha \subset V_{1/\alpha}(0,1)$, it follows from this theorem that:

Theorem 5′. 1) If the function $f(t) \in \operatorname{Lip} \alpha$ for some $0 < \alpha \leqslant 1$, then for every $\beta > \dfrac{2}{2\alpha+1}$ and $\gamma < \alpha$, (3) holds.

2) For every $\alpha \in (0,1]$ there exists a function $\varphi(t) \equiv \varphi_\alpha(t) \in \operatorname{Lip} \alpha$ for which (3) is not satisfied when $\beta = \dfrac{2}{2\alpha+1}$ and $\gamma = \alpha$.

Thus, Theorem 5′ shows that for series with respect to the Haar system there is an analogue of the theorems of Szasz–Hardy–Littlewood (see ([3]), pp. 139–140 and 145).

A special case of Theorem 5 ($p = 1$) is also P. L. Ul' yanov' s theorem on Fourier coefficients with respect to the Haar system for functions of bounded variation (see ([5]), Theorem 5′).

From Theorem 5 we also obtain

Corollary 3. 1) If $f(t) \in V_p(0,1)$ for some $1 \leqslant p < 2$, then

$$\sum_{n=1}^{\infty} |c_n(f)| < \infty. \tag{4}$$

2) For $p \geqslant 2$ the assertion ceases to be valid.

This corollary is a strengthening of the following theorem of Ciesielski and Musielak ([7]):

If $f(t) \in V_p(0,1)$ for some $1 \leqslant p < 2n$ and

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega^{1-p/2}\left(\frac{1}{n}, f\right) < \infty, \tag{5}$$

then (4) holds.

As Corollary 3 shows, condition (5) in this theorem is superfluous.

§ 3. P. L. Ul' yanov ([5]) showed that for every continuous function $f(t)$ one always has

$$|c_n(f)| \leqslant \frac{\omega(1/n, f)}{\sqrt{n}},$$

and this estimate is sharp in order.

Hence, in particular, it follows that for continuous functions

$$c_n(f) = o\left(\frac{1}{\sqrt{n}}\right).$$

If, however, one turns to the class of absolutely continuous functions, then the following is true.

Theorem 6. 1) For every absolutely continuous function $f(t) \not\equiv \text{const}$ we have

$$c_n(f) \neq o\left(\frac{1}{n\sqrt{n}}\right),$$

2) There exists an absolutely continuous function $f_1(t) \not\equiv \text{const}$ for which

$$c_n(f_1) = O\left(\frac{1}{n\sqrt{n}}\right).$$

3) For every $\varepsilon > 0$ there exists an absolutely continuous function $f_2(t)$ for which

$$c_n(f_2) \neq O\left(\frac{1}{n^{1/2+\varepsilon}}\right).$$

§ 4. In this paragraph we formulate two theorems on functions with nonnegative and monotone Fourier coefficients.

Theorem 7. In order that a continuous function $f(t)$ have nonnegative Fourier coefficients $c_n(f)$ for $n \geqslant 2$, it is necessary and sufficient that it be nondecreasing.

Theorem 8. In order that a function $f(t) \not\equiv \text{const}$ with continuous derivative have monotonically decreasing Fourier coefficients $c_n(f)$ for $n \geqslant 2$, it is necessary and sufficient that the following two conditions be satisfied simultaneously:

1) $f'(t)$ is nondecreasing and negative;
2) $2^{-3/2} \leqslant \dfrac{f'(t)}{f'(x)} \leqslant 2^{3/2}$ for all $0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant 1$.

The bounds in condition 2) are sharp.

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Note: Figure translations are in progress. See original paper for figures.