ON SOME NEW INTEGRAL REPRESENTATIONS OF ANALYTIC FUNCTIONS

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Abstract

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ON SOME NEW INTEGRAL REPRESENTATIONS OF ANALYTIC FUNCTIONS

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New representations of analytic functions are established in the article in the form of Cauchy-type integrals whose densities have a special form. The results are used for constructing bases in certain classes of analytic functions.

In what follows, unless otherwise specified, Γ denotes a simple smooth closed contour dividing the plane into two domains: the interior D^+ and the exterior D^-.

We introduce the following classes of functions:

1. $H(\Gamma)$ is the class of functions of the points $t = x + iy$ of the contour Γ satisfying on t the Hölder condition with some exponent α $(0 < \alpha \leqslant 1)$.
2. $H(D^{(\pm)})$ is the class of functions analytic in the domain $D^{(\pm)}$ and continuous in its closure $\overline{D}^{(\pm)}$, whose boundary values on Γ belong to the class $H(\Gamma)$.
3. $E_p(D^{(\pm)})$ is the class of functions $f(z)$, analytic in the domain $D^{(\pm)}$, for which the inequality
$$\int_{\gamma_r} |f(z)|^p |dz| < C, \qquad p > 1,$$
is satisfied, where γ_r is the image of the circle $|w| = r$ under a conformal mapping of the disk $|w| < 1$ onto the domain $D^{(\pm)}$ (C depends on $f(z)$, but does not depend on r).
4. $A(D^{(\pm)})$ is the class of functions analytic in the domain $D^{(\pm)}$.

Further, by $G(\tau)$ we denote a prescribed function of the points of the contour Γ, belonging to the class $H(\Gamma)$, $\varkappa = \operatorname{ind}_\Gamma G(\tau)$.

Theorem 1. If $f^+(z) \in H(D^+)$ and $\varkappa \geqslant 0$, then

$$f^+(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi^+(\tau)}{G(\tau)} \frac{d\tau}{\tau - z}, \tag{1}$$

where $\varphi^+(z) \in H(D^+)$ and is determined by the equalities

$$\varphi^+(z) = X^+(z)\left[\frac{1}{2\pi i}\int_\Gamma \frac{f^+(\tau)}{X^-(\tau)}\,\frac{d\tau}{\tau - z} + P_{\varkappa-1}(z)\right],$$

$$X^\pm(z) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \frac{\ln[G(\tau)\tau^{-\varkappa}]}{\tau - z}\,d\tau\right\},$$

$P_{\varkappa-1}(s)$ is a polynomial of degree $\varkappa - 1$ with arbitrary coefficients. If, however, $\varkappa < 0$, then the representation (1) will hold provided the conditions

$$\frac{1}{2\pi i}\int_\Gamma \frac{f^+(\tau)}{X^-(\tau)}\,\tau^{k-1}\,d\tau = 0, \qquad k = 1, 2, \dots, |\varkappa|. \tag{2}$$

are fulfilled.

The theorem remains valid if $f^+(z) \in E_p(D^+)$, and Γ is a Lyapunov curve. In this case $\varphi^+(z) \in E_p(D^+)$.

Proof. Let $\varphi^-(z)$ be the value of the integral in formula (1) for $z \in D^-$; then from (1) there follows the relation $f^+(t) - \varphi^-(t) = \varphi^+(t)/G(t)$, $t \in \Gamma$, which may be regarded as the boundary condition of the Riemann problem in the class of functions satisfying the Hölder condition ([1,2]), or in the class $L_p(\Gamma)$ ([3]), depending on the class to which $f^+(z)$ belongs, $H(D^+)$, or $E_p(D^+)$. Hence we conclude that $\varphi^+(z)$ is indeed determined by formula (2) and belongs to the class $H(D^+)$ if $f^+(z) \in H(D^+)$. If, however, $f^+(z) \in E_p(D^+)$, then $\varphi^+(t) \in L_p(\Gamma)$, and the gluing theorems show that $\varphi^+(z)$ will also belong to the class $E_p(D)^+$. The theorem is proved.

Let the function $\alpha(t)$, $t \in \Gamma$, map one-to-one and with preservation of the direction of traversal the Lyapunov contour Γ onto some, generally speaking, other Lyapunov contour $\tilde{\Gamma}$, bounding the domain $\widetilde{D}^+$, with $\alpha'(t) \neq 0$ and $\alpha'(t) \in H(\Gamma)$.

Theorem 2. If $f^+(z) \in H(D^+)$ $(E_p(D^+))$ and $\varkappa \geq 0$, then

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi^+[\alpha(\tau)]}{G(\tau)}\,\frac{d\tau}{\tau - z}, \tag{3}$$

where $\varphi^+(z) \in H(\widetilde{D}^+)$ $(E_p(\widetilde{D}^+))$. For $\varkappa = 0$, $\varphi^+(z)$ is determined by $f^+(z)$ uniquely; if $\varkappa > 0$, then $\varphi^+(z)$ contains linearly $\varkappa$ arbitrary complex constants.

Proof. The justification of the representation (3) reduces to the study of the solvability of the problem of linear conjugation with shift

$$f^+(t) - \varphi^-(t) = \frac{\varphi^+[\alpha(t)]}{G(t)}, \qquad t \in \Gamma, \quad \alpha(t) \in \tilde{\Gamma}, \tag{4}$$

studied in ([4]). In that work it is proved that problem (4) reduces to an analogous problem in the case when the contour is the circle Γ_0, and the unknown functions are analytic inside and outside Γ_0, respectively, while $\alpha(t)$ maps Γ_0 onto itself and has the properties indicated above.

We then argue as in the proof of Theorem 1, using the results of ([5,6]).

Similarly one proves

Theorem 3. If Γ, $\tilde{\Gamma}$, and $\alpha(t)$ satisfy the conditions of Theorem 2 and $f^-(z) \in H(D^-)$ $(f^-(\infty) = 0)$, then for $\varkappa \geq 1$

$$f^-(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\varphi^+[\alpha(\tau)]}}{G(\tau)} \frac{d\tau}{\tau - z}, \tag{5}$$

where $\varphi^+(z) \in H(\widetilde{D}^+)$ $(E_p(\widetilde{D}^+))$. For $\varkappa = 1$, $\varphi^+(z)$ is determined uniquely, while for $\varkappa > 1$ it depends linearly on $\varkappa - 1$ arbitrary complex constants.

Let us now consider a function $G(z)$ analytic and different from zero in the curvilinear annulus K bounded by the closed Jordan curves Γ and Γ_1 (Γ_1 lies inside Γ). Clearly, the index of the function $G(z)$ on any contour lying inside K and enclosing Γ_1 will be one and the same. Consequently, one may speak of the index $\varkappa$ of the function $G(z)$ in the annulus K: $\varkappa = \operatorname{ind}_K G(z)$.

Theorem 4. If $f^+(z) \in A(D^+)$, then for $\varkappa \geq 0$

$$f^+(z) = \frac{1}{2\pi i} \int_\gamma \frac{\varphi^+(\tau)}{G(\tau)} \frac{d\tau}{\tau - z}, \tag{6}$$

where $\varphi^+(z) \in A(D^+)$ and depends linearly on $\varkappa$ arbitrary real constants, γ is an arbitrary rectifiable contour lying in the annulus K and enclosing Γ_1.

The proof of Theorem 4 follows immediately from the analysis of the relation $\varphi^+(t) = -G(t)\varphi^-(t) + f^+(t)G(t)$, $t \in \gamma$, equivalent to equality (6).*

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* In Theorems 2 and 4 one could also have considered the case $\varkappa < 0$, and in Theorem 3 the case $\varkappa < 1$, by analogy with Theorem 1.

Below we consider bases in the classes of functions $H(D^{(\pm)})$, $E_p(D^{(\pm)})$, $A(D^{(\pm)})$. By a basis in the class $H(D^{(\pm)})$ $(E_p(D^{(\pm)}), A(D^{(\pm)}))$ we mean a system of functions belonging to the corresponding class such that every function of the class under consideration is uniquely expanded in the series

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z), \tag{7}$$

where, if the class $A(D^{(\pm)})$ $(H(D^{(\pm)}))$ is taken, the series (7) must converge uniformly in the domain $D^{(\pm)}$ (the closed domain $\overline{D^{(\pm)}}$); if, however, the class $E_p(D^{(\pm)})$ is considered, then convergence of the series (7) on the contour Γ in the norm of the space $L_p(\Gamma)$ must hold.

With the aid of Theorems 1—4 the following propositions are easily proved.

Theorem 5. If $\{\varphi_k^+(z)\}$ is a basis in $H(D^+)$ $(E_p(D^+))$, then:

a) the functions

$$f_k^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi_k^+(\tau)}{G(\tau)}\,\frac{d\tau}{\tau - z}$$

for $\varkappa = 0$ also form a basis in $H(D^+)$ $(E_p(D^+))$;

b) for $\varkappa > 0$, any function from $H(D^+)$ $(E_p(D^+))$ can be expanded in the series (7), but the expansion may fail to be unique;

c) for $\varkappa < 0$, the functions $f_k^+(z)$ form a basis among the functions of the class under consideration that satisfy condition (2).

For brevity of exposition, in the theorems below we restrict ourselves to the cases $\varkappa = 0$ and $\varkappa = 1$.

Theorem 6. If $\{\varphi_k^+(\xi)\}$ is a basis in $H(\widetilde{D}^+)$ $(E_p(\widetilde{D}^+))$ and $\varkappa = 0$, then the functions

$$f_k^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi_k^+[\alpha(\tau)]}{G(\tau)}\,\frac{d\tau}{\tau - z}$$

also form a basis in $H(\widetilde{D}^+)$ $(E_p(\widetilde{D}^+))$.

Theorem 7. If $\{\varphi_k^+(\xi)\}$ is a basis in $H(\widetilde{D}^+)$ $(E_p(\widetilde{D}^+))$ and $\varkappa = 1$, then the functions

$$f_k^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\varphi_k^+[\alpha(\tau)]}}{G(\tau)}\,\frac{d\tau}{\tau - z}$$

form a basis among the functions of the class $H(D^-)$ $(E_p(D^-))$ that vanish at infinity.

Theorem 8. If $\{\varphi_k^+(z)\}$ is a basis in the class $A(D^+)$, then for $\varkappa = 0$ the functions

$$f_k^+(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi_k^+(\tau)}{G(\tau)(\tau - z)}\,d\tau,$$

where γ is any rectifiable curve lying inside the ring K, also form a basis in $A(D^+)$.

We note that, in the proof of Theorems 5–7 in the part concerning the class $H(D^+)$, the following proposition plays an essential role:

Lemma. If $\varphi_k^+(z) \in H(D^+)$, $k = 1, 2, ...$, $a(t) \in H(\Gamma)$, and $\lim\limits_{k\to\infty} \max |\varphi_k(t)| = 0$, then $\lim\limits_{k\to\infty} \max |h_k(t)| = 0$, where

$$h_k(t) = \frac{1}{\pi i} \int_\Gamma \frac{a(\tau)\varphi_k^+(\tau)}{\tau - t}\, d\tau, \qquad t \in \Gamma.$$

Below we formulate some corollaries of Theorems 5-7. Let $\Delta^-(z)$ be a given function belonging to the class $H(D^-)$ and different from zero in $\overline{D}^-$.

Corollary 1 (see Theorem 5). Let $P_k(z)$, $k = 1, 2, ...$, be a polynomial basis in $H(D^+)$. Then the functions $Q_k(z)$—the collection of the terms with nonnegative powers of z in the Laurent expansion of the functions $\Delta^-(z)P_k(z)$ in a neighborhood of the point $z = \infty$—will form a basis.

Let $\Phi_0(z)$ be the function mapping D^- onto the exterior of the unit disk; $\Phi_1(z)$, the function mapping D^- onto $\widetilde{D}^-$; and $\Phi_k(z)$ ($\widetilde{\Phi}_k(z)$), the Faber polynomials for the domain D^+ ($\widetilde{D}^+$).

Corollary 2 (see Theorem 6). Each of the systems of functions

$$^{(1)}\Psi_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi_0^k(\tau)}{G(\tau)} \frac{d\tau}{\tau - z}, \qquad ^{(2)}\Psi_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\widetilde{\Phi}_k[\Phi_1(\tau)]}{G(\tau)} \frac{d\tau}{\tau - z}$$

forms a basis in the class $H(D^+)$ ($E_p(D^+)$, if Γ is a Lyapunov curve).

Corollary 3 (see Theorem 7). Each of the systems of functions

$$^{(1)-}\Omega_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi_0^k(\tau)}{G(\tau)} \frac{d\tau}{\tau - z}, \qquad ^{(2)-}\Omega_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\widetilde{\Phi}_k[\Phi_1(\tau)]}{G(\tau)} \frac{d\tau}{\tau - z},$$

$$^{(3)-}\Omega_k(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\Phi_k(\tau)}}{G(\tau)} \frac{d\tau}{\tau - z} \qquad (\chi = 1)$$

forms a basis among the functions of the class $H(D^-)$ ($E_p(D^-)$, if Γ is a Lyapunov curve) that vanish at infinity.

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Note: Figure translations are in progress. See original paper for figures.