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Abstract

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ON THE SOLVABILITY OF THE CAUCHY PROBLEM FOR EVOLUTION EQUATIONS

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The paper studies the equation

$$\frac{d^n u(t)}{dt^n} + A_1(t)\frac{d^{n-1}u(t)}{dt^{n-1}} + \dots + A_n(t)u(t) = f(t) \tag{1}$$

with initial conditions

$$u^{(i)}(0) = u_i \qquad (i = 0, \dots, n-1), \tag{2}$$

where $u(t)$ is the unknown function with values in a Banach space E; $A_i(t)$ are operators acting in E.

The investigation of problem (1)–(2) is carried out by methods of the theory of semigroups. By a **solution of problem** (1)–(2) we shall mean a function $u(t)$, $(n-1)$ times continuously differentiable on $[0,T]$, n times continuously differentiable on $(0,T]$, satisfying equation (1) for every $t \in (0,T]$ and the initial conditions (2), and possessing, in addition, the property that the functions $A_i(t)d^{n-i}u(t)/dt^{n-i}$ $(i = 1, \dots, n)$ and $A_1(t)d^i u(t)/dt^i$ $(i = 1, \dots, n-2)$ are continuous on $(0,T]$.

1. Let us first consider the case of a first-order equation

$$\frac{du(t)}{dt} = Au(t) + f(t) \tag{3}$$

with the initial condition

$$u(0) = u_0. \tag{4}$$

Problem (3)–(4), when A generates a semigroup of class C_0, was studied by R. Phillips (2). The case when A is a strongly positive operator, i.e. generates a semigroup of class $H(\Phi_1, \Phi_2)$, was considered by M. Z. Solomyak (3) and K. Yosida (4). Problem (3)–(4) with a variable operator $A(t)$ has been studied in detail in the works of T. Kato ([5]), M. A. Krasnosel' skii, S. G. Krein, and P. E. Sobolevskii ([6]), P. E. Sobolevskii ([7]), and others. The case in which $A(t)$, for every $t \in [0, T]$, is a generating operator of class C_0 or $(C_0)_u$ has been studied in detail ([1]).

Lemma. Let A be the generating operator of some semigroup $T(t)$ of class $(0, A)$ $[(1, A)]$*. If $f(t)$ is continuously differentiable (continuous) on $[0, \infty)$, then

$$g(t) = \int_0^t T(t-\tau) f(\tau)\, d\tau$$

is continuous on $[0, \infty)$. If, however, $f(t)$ is twice continuously differentiable (continuously differentiable), then $g(t)$ is continuously differentiable and, for $t > 0$,

$$g'(t) = T(t)f(0) + \int_0^t T(t-\tau) f'(\tau)\, d\tau = f(t) + A \int_0^t T(t-\tau) f(\tau)\, d\tau.$$

* In ([1]), a generating operator is called an infinitesimal generating operator.

Let us note that for semigroups of class C_0 this lemma was proved by R. Phillips ([2]).

Theorem 1. Suppose that the following conditions are satisfied: 1) the closed linear operator A is the infinitesimal generator of some semigroup $T(t)$ of class $(0, A)$ $[(1, A)]$; 2) $f(t)$ is a twice continuously differentiable (continuously differentiable) vector-function; 3) $u_0 \in D(A)$.

Then the solution of problem (3)–(4) exists and is given by the formula

$$u(t) = T(t)u_0 + \int_0^t T(t-\tau) f(\tau)\, d\tau, \tag{5}$$

and the equality

$$\frac{du(t)}{dt} = AT(t)u_0 + T(t)f(0) + \int_0^t T(t-\tau) f'(\tau)\, d\tau$$

holds.

If, moreover, $u_0 \in D(A^2)$ and $f(0) \in D(A)$, then equation (3) is satisfied also for $t = 0$.

Remark. If $u_0 \in E$, then formula (5) will be called a **generalized solution of equation** (3). The expediency of this definition follows from Theorem 1. We replace the initial condition (4) by the condition

$$\lim_{\lambda \to \infty} \lambda \int_0^T e^{-\lambda t} u(t)\, dt = u_0 \tag{6}$$

(the continuity of $u(t)$ at zero in the Abel sense). It is easy to see that the generalized solution of problem (3)–(6) is continuous on $(0, T]$.

2. In the work of B. S. Mityagin ([8]) the equation

$$\frac{d^n u(t)}{dt^n} + A_1 \frac{d^{n-1} u(t)}{dt^{n-1}} + \cdots + A_{nu}(t) = f(t) \tag{7}$$

with operators A_i independent of t ($i = 1, \dots, n$) was investigated. In the case $n = 2$, problem (1)–(2) was investigated by P. E. Sobolevskii ([9,10]) and A. Balakrishnan ([11]).

In the present article the method of reducing a higher-order equation to a system is used (this method differs from the method developed in ([8–10])). For brevity we shall illustrate the method for $n = 2$:

$$\frac{d^2 u(t)}{dt^2} + A(t) \frac{du(t)}{dt} + B(t) u(t) = f(t), \tag{8}$$

$$u(0) = u_0, \qquad u'(0) = u_1.$$

With the aid of the substitution $v(t) = du(t)/dt$ and $w(t) = A(0)u(t)$, problem (8) is reduced to the Cauchy problem for a system of first-order evolutionary equations

$$\frac{dw(t)}{dt} - A(0)v(t) = 0,$$

$$\frac{dv(t)}{dt} + A(t)v(t) + B(t)A^{-1}(0)w(t) = f(t), \tag{9}$$

$$w(0) = A(0)u_0,$$

$$v(0) = u_1.$$

In the topological product $E \times E$, problem (9) can be written in the form

$$\frac{dU(t)}{dt} + \mathfrak{A}(t)U(t) = F(t),$$

$$U(0) = U_0,$$

where

$$U(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad U_0 = \begin{pmatrix} A(0)u_0 \\ u_1 \end{pmatrix},$$

$$\mathfrak{A}(t) = \begin{pmatrix} 0 & -A(0) \\ B(t)A^{-1}(0) & A(t) \end{pmatrix} = \begin{pmatrix} 0 & -A(0) \\ 0 & A(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B(t)A^{-1}(0) & 0 \end{pmatrix} = \mathfrak{A}_1(t) + \mathfrak{A}_2(t).$$

In the topological product $E \times E$ the resolvent of the operator $\mathfrak{A}_1(t)$ is expressed in terms of the resolvent of the operator $A(t)$:

$$R(\lambda, -\mathfrak{A}_1(t)) = \begin{pmatrix} \frac{1}{\lambda} A(0)A^{-1}(t) \left[\frac{1}{\lambda} - R(\lambda, -A(t))\right] \\ 0 \quad R(\lambda, -A(t)) \end{pmatrix}. \tag{10}$$

Formula (6) makes it possible to express any function of the operator $\mathfrak{A}_1(t)$ in terms of the same function of the operator $A(t)$; namely, the formula

$$f(\mathfrak{A}_1(t)) = \begin{pmatrix} \beta & A(0)A^{-1}(t)\,[\beta - f(A(t))] \\ 0 & f(A(t)) \end{pmatrix},$$

is proved, where β is a certain number depending on the function $f(z)$.

Applying perturbation theory for various classes of semigroups and Theorem 1, we prove Theorem 2.

Theorem 2. *Suppose the following conditions are satisfied: 1) the closed linear operator $-A_1$ is the infinitesimal generator of a certain semigroup of class $(0, A)$ $[(1, A)]$; 2) the operators $A_{iA}1^{-1}$ $(i = 1, \dots, n)$ are bounded; 3) $f(t)$ is a twice continuously differentiable (continuously differentiable) vector function; 4) $u_i \in D(A)$ $(i = 0, \dots, n)$.*

Then problem (1)–(2) has a solution, and moreover a unique one.

In the analogous theorem of B. S. Mityagin ((8), Theorem 4), under the assumption that $-A_1$ is the infinitesimal generator of a semigroup of class C_0, more is required, namely, the possibility of the decomposition

$$\sum_{j=1}^{n} A_j \lambda^{n-j} = A_1(\lambda - F_1)(\lambda - F_2)\cdots(\lambda - F_{n-1}),$$

where F_i $(i = 1, \dots, n-1)$ are bounded operators, and their spectra are pairwise disjoint.

Theorem 3. *Suppose the following conditions are satisfied: 1) the operator* $A_1(t)$ $(t \in [0, T])$ *acts in* E*, has a domain of definition everywhere dense and independent of* t*, and the estimate* $\|[A_1(t) + \lambda I]^{-1}\| \le \dfrac{1}{\lambda + 1}$ *holds* $(\lambda > -1)$*; 2) the operator-functions* $A_i(t)A_1^{-1}(0)$ $(i = 1, \dots, n)$ *are once strongly continuously differentiable on* $[0, T]$*; 3) the vector function* $f(t)$ *is continuously differentiable on* $[0, T]$*; 4)* $u_i \in D$ $(i = 0, \dots, n-1)$*.*

Then problem (1)–(2) has a solution, and moreover a unique one.

Applying the results of (7), one can formulate an analogous theorem when the operator $A_1(t)$ is strongly positive. For $n = 2$, Theorem 2 was proved by P. E. Sobolevskii (10) under twice continuous differentiability of the operator function $A_1(t)A_1^{-1}(0)$.

3. We turn to consideration of the nonlinear equation

$$\frac{d^n u(t)}{dt^n} + A_1(t)\frac{d^{n-1}u(t)}{dt^{n-1}} + \cdots + A_n(t)u(t) = f(t, u(t)). \tag{11}$$

For $n = 2$, in the work (10) a local existence theorem was proved for the solution of problem (11)–(2). The following theorem is the n-dimensional analogue of this theorem. Denote by S_0 a certain ball of the space E with center at $A(0)u_0$.

Theorem 4. *Suppose conditions 1), 2), and 4) of Theorem 2 are satisfied. Suppose the operator* $f(t, A^{-1}(0)u)$ *on* $[0, T] \times S_0$ *has continuous partial derivatives with respect to the aggregate of variables* $f'_t(t, A^{-1}(0)u)$, $f'_u(t, A^{-1}(0)u)$ *(the derivative is understood in the sense of Fréchet), satisfying in* u *a Lipschitz condition (*$f'_u(t, A^{-1}(0)u)$ *with respect to the norm of the space of linear operators over* E*).*

Then there exists, and moreover is unique, a solution of problem (11)–(2) on some interval $[0, t_0] \subset [0, T]$*.*

4. In some problems of mathematical physics one encounters the partial differential equation [12]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial t}\Delta u - \Delta u = f(x, y, z, t) \qquad \left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right). \tag{12}$$

In [12], formal solutions are constructed for the Cauchy problem (for $t = 0$, $u = u_0(x, y, z)$, $\partial u/\partial t = u_1(x, y, z)$) and for one mixed problem (for $t = 0$,

$u = u_0(x, y, z)$, $\partial u/\partial t = u_1(x, y, z)$; for $z = 0$, $\partial u/\partial z = 0$). Applying the results of article [13] and of the present article, one can obtain existence theorems for equations more general than [12].

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Note: Figure translations are in progress. See original paper for figures.