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**Abstract**

**Full Text**

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## ON OSCILLATIONS OF A VISCOUS LIQUID IN A VESSEL

(Presented by Academician A. Yu. Ishlinskii, 26 V 1964)

**Statement of the problem.** The paper considers the question of motions of a heavy viscous incompressible liquid in an open vessel, close to the equilibrium position. Surface tension is not taken into account. Linearization of the Navier–Stokes equations leads to the system consisting of the equations

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u} - \nabla p_1, \quad \operatorname{div} \mathbf{u} = 0 \quad (1)$$

in the region  $\mathcal{G}$  filled with liquid, the boundary conditions  $\mathbf{u} = 0$  on the wall  $\Gamma_1$  of the vessel, and

$$\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \quad \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \quad \frac{\partial}{\partial t} \left( p_1 - 2\nu \frac{\partial u_z}{\partial z} \right) = g u_z \quad (2)$$

on the free surface of the liquid  $\Gamma_0$ .

Here  $\nu = \mu/\rho$  is the kinematic coefficient of viscosity,  $p_1 = p/\rho + gz$ , where  $p$  is pressure,  $\rho$  is the density of the liquid,  $g$  is the acceleration of gravity, and  $\mu$  is the coefficient of viscosity.

**Main results.**

**Theorem 1.** *There exists a unique generalized solution of equations (1)–(2), satisfying the prescribed initial condition  $\mathbf{u}(0, x, y, z) = \mathbf{u}_0(x, y, z) \in \widetilde{W}_2^1$ .*

The class of generalized solutions and the space  $\widetilde{W}_2^1$  will be described below.

**Normal oscillations** are solutions of (1)–(2) for which

$$\mathbf{u}(t, x, y, z) = e^{-\lambda t} \mathbf{v}(x, y, z), \quad p_1(t, x, y, z) = e^{-\lambda t} q_1(x, y, z).$$

**Theorem 2.** *There exists a countable number of normal oscillations. All normal oscillations are aperiodic motions, with the possible exception of a finite number of damped oscillations. There exist arbitrarily rapidly decaying aperiodic motions ( $\lambda \rightarrow \infty$ ) and arbitrarily slowly decaying ones ( $\lambda \rightarrow 0$ ). For sufficiently large viscosity, there are no oscillatory motions.*

The problem of oscillations of a liquid is of practical interest in connection with the oscillations of solid bodies having cavities with liquids. If, in the scheme of an ideal liquid, resonance phenomena are possible at arbitrarily high frequencies of oscillations of the bodies, then for a viscous liquid, as Theorem 2 shows, these phenomena can be observed only in a bounded range of frequencies, and, for large viscosity, are altogether impossible.

The condition for the absence of oscillatory motions of the liquid can be written in the form

$$\nu^2 \geq 4gab, \quad (3)$$

where  $a$  and  $b$  are constants determined by the geometric properties of the region filled with the liquid. If the latter condition is not fulfilled, then for oscillatory motions  $\mathbf{u} = e^{-\alpha t} \sin(\beta t + \varphi) \mathbf{v}$

$$\frac{\nu}{2a} \leq \sqrt{\alpha^2 + \beta^2} \leq \frac{2gb}{\nu}. \quad (4)$$

We shall describe the course of the arguments by means of which Theorems 1 and 2 are obtained. Two auxiliary problems are considered.

Equations of problem I:

$$-\nu \Delta \mathbf{s} + \nabla p_2 = \mathbf{f}, \quad \operatorname{div} \mathbf{s} = 0,$$

where  $\mathbf{f}$  is a given vector function; the boundary conditions are

$$\mathbf{s} = 0 \quad \text{on } \Gamma_1,$$

$$\frac{\partial s_x}{\partial z} + \frac{\partial s_z}{\partial x} = 0, \quad \frac{\partial s_y}{\partial z} + \frac{\partial s_z}{\partial y} = 0, \quad -p_2 + 2\nu \frac{\partial u_z}{\partial z} = \varphi \quad \text{on } \Gamma_0.$$

The equations of problem II:

$$-\nu \Delta \mathbf{w} + \nabla p_3 = 0, \quad \operatorname{div} \mathbf{w} = 0,$$

boundary conditions:

$$\mathbf{w} = 0 \quad \text{on } \Gamma_1,$$

$$\frac{\partial w_x}{\partial z} + \frac{\partial w_z}{\partial x} = 0, \quad \frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y} = 0, \quad -p_3 + 2\nu \frac{\partial w_z}{\partial z} = \varphi \quad \text{on } \Gamma_0,$$

where  $\varphi$  is a function given on  $\Gamma_0$ .

The formulation of these two problems is refined and generalized. By  $\widetilde{W}_2^1$  we denote the set of all solenoidal vector functions  $\mathbf{v}(x, y, z)$ , whose components belong to the Sobolev space  $W_2^1$  (have first derivatives square-summable over the domain  $\mathcal{G}$ ) and satisfy the conditions  $v_x = v_y = v_z = 0$  on  $\Gamma_1$ . In the space  $\widetilde{W}_2^1$  we introduce the scalar product

$$\begin{aligned}
 E(\mathbf{u}, \mathbf{v}) = & \iiint_{\mathcal{G}} \left[ 2 \frac{\partial u_x}{\partial x} \frac{\partial v_x}{\partial x} + 2 \frac{\partial u_y}{\partial y} \frac{\partial v_y}{\partial y} + 2 \frac{\partial u_z}{\partial z} \frac{\partial v_z}{\partial z} + \right. \\
 & + \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \left( \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) + \\
 & \left. + \left( \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \right] d\mathbf{x}.
 \end{aligned}$$

The physical meaning of the expression  $E(\mathbf{u}, \mathbf{v})$  is that it is proportional to the work performed per unit time by the dissipative forces arising in the liquid during its motion, determined by the velocity field  $\mathbf{u}(x, y, z)$ , on a virtual displacement determined by the velocity field  $\mathbf{v}(x, y, z)$ .

The space  $\widetilde{W}_2^1$  is complete with respect to the norm determined by the scalar product  $E(\mathbf{u}, \mathbf{v})$ , which follows from the well-known Korn inequality (see (1)).

A **generalized solution of problem I** is a vector function  $\mathbf{s} \in \widetilde{W}_2^1$  satisfying, for any  $\mathbf{v} \in \widetilde{W}_2^1$ , the identity

$$\nu E(\mathbf{s}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad (5)$$

where  $(\mathbf{f}, \mathbf{v})$  denotes the scalar product in  $\mathcal{L}_2(\mathcal{G})$ :

$$(\mathbf{f}, \mathbf{v}) = \iiint_{\mathcal{G}} \mathbf{f} \cdot \mathbf{v} d\tau.$$

A **generalized solution of problem II** is a vector function  $\mathbf{w} \in \widetilde{W}_2^1$  satisfying, for any  $\mathbf{v} \in \widetilde{W}_2^1$ , the identity

$$\nu E(\mathbf{w}, \mathbf{v}) = (\varphi, v_z)_0, \quad (6)$$

where  $(\varphi, v_z)_0$  denotes the scalar product in  $\mathcal{L}_2(\Gamma_0)$ :

$$(\varphi, v_z)_0 = \iint_{\Gamma_0} \varphi v_z d\sigma.$$

It follows from Green's formula that the classical solutions of problems I and II are, respectively, solutions of problems (5) and (6).

The existence of solutions of problems (5) and (6) is easily proved by means of the Riesz theorem on the general form of a linear functional in a Hilbert space, if one uses the inequalities

$$(\mathbf{v}, \mathbf{v}) \leq aE(\mathbf{v}, \mathbf{v}), \quad (v_z, v_z) \leq bE(\mathbf{v}, \mathbf{v}), \quad (7)$$

which follow from Korn's inequality and the embedding theorems of S. L. Sobolev. Here it should be noted that Korn's inequalities were proved by Friedrichs for domains with sufficiently smooth boundary. In the case under consideration the domain certainly has corner points where the free surface adjoins the wall of the vessel (the effect of wetting or non-wetting is not taken into account). However, in this case as well inequalities (7) are valid, as follows from recent results of Campanato (2).

Denote by  $A_0^{-1}$  and  $T^{-1}$  the operators giving the solution of problems I and II:

$$\nu \mathbf{s} = A_0^{-1} \mathbf{f}, \quad \nu \mathbf{w} = T^{-1} \varphi.$$

The operator  $A_0^{-1}$  is a completely continuous positive self-adjoint operator in  $\mathcal{L}_2(\mathcal{G})$ . In what follows we shall need the square root of the operator  $A_0^{-1}$ —the operator  $A_0^{-1/2}$ .

**Nonstationary problem. A generalized solution of equations (1)–(2)** is a vector-function  $\mathbf{u}(t, x, y, z)$  representable in the form

$$\mathbf{u}(t, x, y, z) = \mathbf{s}(t, x, y, z) + \mathbf{w}(t, x, y, z),$$

where  $\mathbf{s}, \mathbf{w} \in \widetilde{W}_2^1$  for each  $t$ , and satisfying the equations

$$\nu E'(\mathbf{s}, \mathbf{v}) + \left( \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) = 0, \quad (8)$$

$$\nu E \left( \frac{\partial \mathbf{w}}{\partial t}, \mathbf{v} \right) + g(u_z, v_z)_0 = 0 \quad (9)$$

for every vector-function  $\mathbf{v} \in \widetilde{W}_2^1$ .

For the investigation it is very useful that pressure does not enter into the definition of the solution of the problem on the motion of the fluid. The application of the principle of virtual displacements, in general, is apparently a universal method for obtaining equations without pressure, which arises as the reaction of ideal constraints in the fluid and, consequently, does not enter into the equation of work.

Using the operators giving the solution of problems I and II, equations (8) and (9) may be written in the form of a system of operator equations

$$\frac{d\mathbf{s}}{dt} + \frac{d\mathbf{w}}{dt} + \nu A_0 \mathbf{s} = 0, \quad (10)$$

$$\nu \frac{d\mathbf{w}}{dt} + gT^{-1}(w_z + s_z) = 0.$$

Make the substitutions

$$\mathbf{u} = A_0^{-1/2} \vec{\zeta}, \quad \mathbf{s} = A_0^{-1/2} \vec{\eta}, \quad \mathbf{w} = A_0^{-1/2} \vec{\zeta}.$$

Then from system (10) we obtain

$$\frac{d\vec{\eta}}{dt} + \nu A_0 \vec{\eta} - \frac{2}{\nu} Q \vec{\eta} - \frac{g}{\nu} Q \vec{\zeta} = 0, \quad (11)$$

$$\frac{d\vec{\zeta}}{dt} + \frac{g}{\nu} Q \vec{\eta} + \frac{g}{\nu} Q \vec{\zeta} = 0,$$

where

$$Q\mathbf{v} = A_0^{1/2} T^{-1}(A_0^{-1/2} \mathbf{v})_z.$$

**Lemma.** The operator  $Q$  is a completely continuous nonnegative self-adjoint operator in  $\mathcal{L}_2(\mathcal{G})$ .

System (11) can be regarded as a single ordinary differential equation in Hilbert space with an abstract elliptic operator consisting of the sum of the self-adjoint operator  $\mathfrak{A}$  and the completely continuous operator  $\mathfrak{B}$ :

$$\mathfrak{A} = \begin{pmatrix} \nu A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{B} = \frac{g}{\nu} \begin{pmatrix} -Q & -Q \\ Q & Q \end{pmatrix}.$$

This equation was studied by P. E. Sobolevskii and the author in paper <sup>(3)</sup>, from whose results it follows that the equation has a solution for arbitrary initial values  $\vec{\eta}_0, \vec{\zeta} \in \mathcal{L}_2(\mathcal{G})$ . It can be shown that such initial data will correspond to an arbitrary  $\mathbf{u} \in \widetilde{W}_2^1$ , whence the assertion of Theorem 1 follows.

We note that by the same method one can also consider a number of nonhomogeneous problems.

**Normal oscillations.** The equations for normal oscillations take the form

$$\nu E(\mathbf{s}_1, \mathbf{v}) = \lambda(\mathbf{u}_1, \mathbf{v}), \quad \lambda \nu E(\mathbf{w}_1, \mathbf{v}) = g(u_{1z}, v_z)_0,$$

where  $\mathbf{u} = e^{-\lambda t} \mathbf{u}_1$  and  $\mathbf{u}_1 = \mathbf{s}_1 + \mathbf{w}_1$ .

In operator form the oscillation equations are written as follows:

$$\nu \vec{\xi}_1 = \lambda A_0^{-1} \vec{\xi}_1 + \frac{g}{\lambda} Q \vec{\xi}_1 \quad (u_1 = A_0^{-1/2} \xi_1). \quad (12)$$

Such an equation, depending nonlinearly on the parameter  $\lambda$ , was investigated by N. G. Askerov, G. I. Laptev, and the author in <sup>(4)</sup>. The results obtained there imply Theorem 2. In <sup>(4)</sup> it was shown that, under the condition

$$4g \|A_0^{-1}\| \|Q\| \leq \nu^2$$

all eigenvalues are real; otherwise all complex eigenvalues lie in the annulus

$$\frac{\nu}{2\|A_0^{-1}\|} \leq |\lambda| \leq \frac{2g\|Q\|}{\nu}. \quad (13)$$

In the case considered here one may assume that  $\|A_0^{-1}\| = a$  and  $\|Q\| = b$ , whence conditions (3) and (4) follow.

**Completeness of the system of normal oscillations.** In <sup>(4)</sup>, equation (12) was transformed into a system of two equations, already linearly dependent on the parameter. Study of the corresponding operator made it possible to prove a theorem on the completeness of the eigen- and associated solutions of equation (12). However, this theorem was proved under restrictive conditions which are not satisfied in the case considered here. Recently G. I. Laptev <sup>(5)</sup> showed that by another transformation equation (12) is reduced to a system of such a form that the well-known theorem of M. V. Keldysh <sup>(6)</sup> is applicable to it. Thus it turned out that the following is true.

**Theorem 3.** *The normal oscillations and the solutions associated with them form a complete system of vector-functions in the domain  $\mathfrak{G}$ , and their vertical components form a complete system of functions on the free surface  $\Gamma_0$ .*

It should also be noted that associated solutions exist only for a finite number of eigenvalues lying in the annulus (13).

**Bubnov–Galerkin method.** The way in which the concept of a generalized solution was introduced suggests how to choose coordinate functions conveniently for applying the approximate Bubnov–Galerkin method to the problem. It is natural to take as the system of coordinate functions an orthonormal system

in the space  $\widetilde{W}_2^1$ , and to seek  $\mathbf{u}_1$  in the form of a linear combination of these functions; then, for finding the eigenvalues, one obtains the equation

$$\nu\xi = \lambda A\xi + \frac{1}{\lambda}B\xi,$$

where  $\xi$  is a vector, and  $A$  and  $B$  are symmetric positive definite matrices.

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*Note: Figure translations are in progress. See original paper for figures.*

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