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Abstract

Full Text

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UNIVERSAL HOMOTOPY GROUPS

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1. As is well known, the methods of the algebraic theory of homotopy are, in the main, applicable only to spaces having the same homotopy type as some polyhedron (by a **polyhedron** we shall mean in this note a topological space admitting a cell decomposition in the sense of Whitehead). In this, homotopy theory displays a complete analogy with classical homology theory, which is likewise applicable only to polyhedra. The fact that the functor $\pi(X) = \Sigma\pi_n(X)$, assigning to a space X its homotopy groups, was from the very beginning defined by Hurewicz for arbitrary spaces X, does not contradict this in the least, since the properties of the functor π needed for classifying the homotopy type of spaces hold only for polyhedra. In this note we propose a new definition of homotopy groups, suitable for broader categories of spaces and making it possible to obtain for them results as satisfactory as, for example, the results of the theory of Postnikov systems in the category of polyhedra. Our investigations are, in essence, a direct transfer into homotopy theory of methods so successfully developed in homology theory by P. S. Aleksandrov and his school.

Let $\mathfrak{K}$ be an arbitrary subcategory of the category of all normal spaces with a distinguished point and all their mappings taking the distinguished point to the distinguished point. Let, further, $\mathfrak{G}$ be the category of all groups (in general, non-Abelian). We shall say that a certain functor $\Pi : \mathfrak{K} \to \mathfrak{G}$ has **property** W on $\mathfrak{K}$ if an arbitrary mapping $f : X \to Y$ of the category $\mathfrak{K}$ is a homotopy equivalence if and only if the homomorphism $\Pi(f)$ is an isomorphism.

The well-known theorem of Whitehead ([1]), which is one of the most important theorems of the algebraic theory of homotopy, since it more or less directly makes it possible to characterize the homotopy type of any space X, asserts that on the category $\mathfrak{P}$ of all polyhedra with a distinguished point and all their continuous mappings the homotopy functor π has property W. Our aim is to construct, for any category $\mathfrak{K}$, a certain functor $\Pi : \mathfrak{K} \to \mathfrak{G}$ having the following properties: 1) the functor Π has property W on $\mathfrak{K}$; 2) there exists a natural monomorphism $\mu : \pi \to \Pi$, which is an isomorphism when $\mathfrak{K} = \mathfrak{P}$. We shall define this functor by means of a certain system of axioms. Before formulating these axioms, we shall set out the necessary concepts, some of which are apparently of interest in their own right.

2. Let $\Phi : \mathfrak{K} \to \mathfrak{G}$ be an arbitrary functor. A subgroup $G \subset \Phi(X)$, $X \in \mathfrak{K}$, is called **admissible** if there exists a mapping $f : T \to X$, $f \in \mathfrak{K}$, such

that $G \subset \Phi(X)$. The class $\{f\}$ of all such mappings is denoted by the symbol $\overline{G}$, and the set $\{G\}$ of all admissible subgroups of the group $\Phi(X)$ by the symbol $\overline{\Phi}(X)$. It is evident that the set $\overline{\Phi}(X)$ is partially ordered by inclusion. Moreover, it is easy to see that this set is a complete structure, provided only that the following three axioms are satisfied:

Axiom 1 (homotopy axiom). If $f_1 \sim f_2$, then

$$\Phi(f_1) = \Phi(f_2).$$

Axiom 2. Let $K = K_\Phi$ be the class of all maps $k \in \mathfrak{K}$ having the property that the homomorphism $\Phi(k)$ is an epimorphism. Then, for any maps k_ι, $\iota \in I$, of the class K, their bouquet sum $\bigvee_{\iota \in I} k_\iota$ also belongs to the class K.

Axiom 3. For subgroups $G_i \in \overline{\Phi}(X)$, $i = 1, 2$, the inclusion $G_1 \subset G_2$ holds if and only if, for any maps $f_i \in \overline{G_i}$, there exist maps $r \in \mathfrak{K}$ and $k \in K$ such that $f_1 k \sim f_2 r$.

The operation $\vee$ in $\overline{\Phi}(X)$ is then defined by the formula

$$G_1 \vee G_2 = \Phi(\varphi \circ (f_2 \vee f_2))\, \Phi(T_1 \vee T_2), \qquad G_1, G_2 \in \overline{\Phi}(X),$$

where $f_i : T_i \to X$, $i = 1, 2$, are arbitrary maps from $\overline{G_i}$, and $\varphi : X \vee X \to X$ is the "folding" map. It is easy to see that this definition does not depend on the choice of the maps f_i and immediately generalizes to the case of any number of summands. The operation $\wedge$ is then defined by the formula

$$\bigwedge G_\iota = \bigvee H,$$

where H ranges over those subgroups of $\overline{\Phi}(X)$ such that $H \subset G_\iota$ for all $\iota \in I$.

To each map $f : X \to Y$, $f \in \mathfrak{K}$, we assign the map $\overline{\Phi}(f) : \overline{\Phi}(X) \to \overline{\Phi}(Y)$, setting $\overline{\Phi}(f)G = \Phi(f)G$. A simple verification shows that, by virtue of this definition, for any functor $\Phi : \mathfrak{K} \to \mathfrak{G}$ satisfying axioms 1)–3), the operation $\overline{\Phi}$ is a functor $\mathfrak{K} \to \mathfrak{B}$, where $\mathfrak{B}$ is the category of all complete structures and all their maps $V_1 \to V_2$, $V_i \in \mathfrak{B}$, that are $\vee$-homomorphisms and carry the zero (the least element) of the structure V_1 into the zero of the structure V_2.

A map φ of the category $\mathfrak{B}$ will be called an **isomorphism** if it has, in $\mathfrak{B}$, both a left and a right inverse map (as is known, bijectivity of the map is not sufficient for this). A decomposition $a = \bigvee a_\iota$, $\iota \in I$, of an element $a \in V \in \mathfrak{B}$ will be called **proper** if $a \neq \bigvee_{\iota \neq \tau} a_\iota$ for no $\tau \in I$. In this case we shall write $a = \dot{\bigvee} a_\iota$, $\iota \in I$. An element $a \in V$ will be called **whole** if it is either indecomposable, or, for every decomposition $a = \dot{\bigvee} a_\iota$, at least one element a_ι is properly decomposable.

3. We can now list the axioms that the functor Π must satisfy in addition to axioms 1–3.

Axiom 4 (dimension axiom). For a space consisting of a single point x_0, the group $\Phi(X)$ is trivial (consists only of the identity e).

Axiom 5. Let K^* be the class of all maps $k \in \mathfrak{K}$ having the following property: if $gl \sim k$, where $g, l \in \mathfrak{K}$, and moreover, for any map $f \in \mathfrak{K}$, the relation $kf \sim 0$ implies the relation $lf \sim 0$, then the map g has (in $\mathfrak{K}$) a right homotopy inverse. Let, furthermore, K' be the class of all such maps $k' \in \mathfrak{K}$ that every map $k \in K^*$, for which the composition kk' is defined, has (in $\mathfrak{K}$) a right homotopy inverse. Then $K \subset K^*$, and for every nonidentity map $k \in K$ there exists a map $k_1 \in K$ such that $k_1 k \in K'$.

Axiom 6. For any functor Ψ satisfying axioms 1–5, every map $k \in K_\Phi$ for which the map $\Psi(k)$ is an isomorphism is a homotopy equivalence.

Axiom 7. Every element $a \in \Phi(X)$ belongs to the subgroup

$$U(a) = \bigwedge B,$$

where B ranges over all subgroups of $\overline{\Phi}(X)$ for which $a \in B$.

Axiom 8. Any proper decomposition $\bigvee A_t$ of the subgroup $U(a)$ is finite, and in the subgroups A_t there exist elements $a_t \in \Phi(X)$ such that $a = \sum a_t$.

Axiom 9. There exists a natural monomorphism $\mu : \pi \to \Phi$ (where π, as above, is the ordinary homotopy functor), which is an isomorphism when $\mathfrak{K} = \mathfrak{P}$.

Axiom 10. The functor Φ is universal with respect to axioms 1-9, i.e. for any functor $\Phi' : \mathfrak{K} \to \mathfrak{G}$ satisfying these axioms, there exists a natural epimorphism $\varphi : \Phi' \to \Phi$.

Theorem. *There exists one and only one functor* $\Pi : \mathfrak{K} \to \mathfrak{G}$ *satisfying axioms 1-10. This functor has property* W *on* $\mathfrak{K}$.

We shall prove this theorem in two stages.

4. First of all we shall construct a functor $\overline{W} : \mathfrak{K} \to \mathfrak{B}$ and prove its uniqueness. Let $f_i : T_i \to X$, $i = 1, 2$, be arbitrary maps of the category $\mathfrak{K}$. We shall say that $f_1 \leq f_2$ if there exists a map $r \in \mathfrak{K}$ such that $f_2 r \sim f_1$, and that $f_1 \equiv f_2$ if $f_1 \leq f_2 \leq f_1$. The relation $\equiv$ is an equivalence relation and partitions the set of all maps $f : T \to X$, $f \in \mathfrak{K}$, into disjoint equivalence classes $\bar{f}$. Let $\overline{W}(X)$ be the set of all these classes. It is partially ordered and is a complete lattice with respect to the operation $\bigvee$ induced by bouquet addition of maps. Putting $\overline{W}(g)\bar{f} = \overline{gf}$ for any $g \in \mathfrak{K}$, we immediately obtain that the correspondence $\overline{W} : \mathfrak{K} \to \mathfrak{B}$ is a functor.

Now let F be some class of maps of the category $\mathfrak{K}$ satisfying the following conditions:

F1) the class F is the totality of all maps of some subcategory of the category $\mathfrak{K}$ containing all spaces of the category $\mathfrak{K}$;

F2) the class F is closed with respect to the operation of bouquet addition of maps and, for any bouquet $\bigvee X_i$ of identical spaces ($X_i = X \in \mathfrak{K}$), contains the corresponding folding map $\bigvee X_i \to X$;

F3) for any maps $f_1 : T_1 \to X_1$, $f_1 \in \mathfrak{K}$, and $k_1 : T_2 \to X$, $k_1 \in F$, there exist maps $f_2 \in \mathfrak{K}$, $k_2 \in F$, such that $f_1 k_2 \sim k_1 f_2$.

We shall say that $f_1 \leq f_2$ rel. F, if there exist maps $k \in F_1$, $r \in \mathfrak{K}$, such that $f_1 k \sim f_2 r$ (cf. axiom 3). Starting from this relation, we, as above, can define a certain functor $\overline{W}_F : \mathfrak{K} \to \mathfrak{B}$, which is a quotient functor of the functor $\overline{W}$. It turns out that among the classes F there exists a class K, maximal with respect to the properties of the classes K indicated in axioms 5 and 6.

This class K has, in addition, the property that the corresponding functor $\overline{W}_K$ satisfies the conditions of Whitehead' s theorem, i.e. a map $g : X \to Y$, $g \in \mathfrak{K}$, is a homotopy equivalence if and only if the map $\overline{W}_K(g)$ is isomorphic (or even merely bijective). Moreover, the functor $\overline{W}_K$ is the unique (up to isomorphism) functor universal with respect to axioms 1-6 (formulated for the functor $\overline{W}_K$). Finally, when $\mathfrak{K} = \mathfrak{P}$, the functor $\overline{W}_K$ coincides with the functor π.

5. Let $\Phi_1, \Phi_2 : \mathfrak{K} \to \mathfrak{G}$ be arbitrary functors satisfying axioms 1-9. We shall say that $\Phi_1 \leq \Phi_2$ if there exists a homomorphism $\Phi_2 \to \Phi_1$ inducing an epimorphism $\Phi_2 \to \Phi_1$. Using the sets $U(a)$ (see axioms 7 and 8), one can easily prove that this relation satisfies the conditions of Zorn's lemma. The corresponding minimal functor Φ is the functor we need. Its uniqueness is proved without difficulty.

6. **Remark.** Since $f_1 \equiv f_2$ if $f_1 \sim f_2$, the relation $\leq$ carries over to homotopy classes. In connection with this relation one can pose a number of interesting problems. For example, let C_f be the cone of some map–

map $f : S^n \to X$, and let $i : X \to C_f$ be the natural inclusion. We pose the question: for every element $a \in \operatorname{Coker} \pi(i)$, does there exist a nonzero element $\beta \in \operatorname{Im} \pi(i)$ such that $\beta \leqslant a$? I do not know the answer to this question even for polyhedra.

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1. J. H. C. Whitehead, Bull. Am. Math. Soc., 213 (1949).

Note: Figure translations are in progress. See original paper for figures.