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Abstract

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MATHEMATICS

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HOROSPHERES IN CERTAIN GRASSMANN MANIFOLDS

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In the present paper we study the geometry on surfaces that are analogues of classical horospheres in certain Grassmann manifolds.

1. Consider a non-Euclidean space ${}^{l}S_{n}$ of index l, which we shall interpret as a metrized projective space with absolute Ω, which is a nondegenerate quadric of index l. Denote by ${}^{l,k}S_{n,m}$ the manifold of m-dimensional planes of the space ${}^{l}S_{n}$ carrying a non-Euclidean geometry of index k $(0 \leq l \leq (n+1)/2,\ 0 \leq k \leq l)$. These manifolds are symmetric spaces of rank $m+1$, Riemannian in the case ${}^{l,l}S_{n,l-1}$ and pseudo-Riemannian in the remaining cases. The elements of the space ${}^{l,k}S_{n,m}$, i.e. the planes of the space ${}^{l}S_{n}$, can be specified by $(n+1) \times (m+1)$-matrices A, whose columns consist of the projective coordinates of the basis points of the plane A. Obviously, the matrix A is specified up to replacing its columns by any linear combinations of them, i.e. up to multiplication on the right by an arbitrary nonsingular $(m+1) \times (m+1)$-matrix. Moreover, since the intersection of the plane A with the absolute must be a quadric of index k, the matrix $Q = A^{T}\Omega A$ (where Ω is the matrix of the equation of the absolute) must be a nonsingular matrix of this index. The matrices A may be normalized by the condition that Q be the diagonal matrix $\mathcal{E}$, containing k entries -1 and $m-k+1$ entries $+1$. Then the matrix A will already be determined up to multiplication on the right by a pseudo-orthogonal matrix of index k.

The space ${}^{l,k}S_{n,m}$ can be completed by improper elements. In ${}^{l}S_{n}$ they will be represented by m-dimensional planes lying on the absolute Ω or tangent to it. In what follows we shall need only the improper elements of the space ${}^{l,k}S_{n,m}$ represented in ${}^{l}S_{n}$ by generators of the absolute. We shall call these elements infinitely distant. They exist for $m < l$, and precisely this case alone will be considered below.

2. We shall say that two elements A and B of the space ${}^{l,k}S_{n,m}$ have a common position if the planes representing them have $m+1$ common perpendiculars and if the feet of these perpendiculars, i.e. the points of their intersections with the planes, are linearly independent*. In this case the feet of distinct perpendiculars are polar-conjugate with respect to the absolute, and if they are taken as the basis points of the planes A and B, then it turns out that $A^T \Omega A = B^T \Omega B = \mathcal{E}$. Moreover, in this case the matrix $W(A, B) = \mathcal{E} B^T \Omega A \, \mathcal{E} A^T \Omega B$ will be diagonal, and its entries will be positive. If we put $W = \operatorname{ch}^2 p$, then the entries of the matrix will be the lengths of the common perpendiculars of the planes A and B (see (^{1})). Obviously, the matrix p is an invariant of the group of motions; it is called the compound distance between A and B. We shall agree to call the elements U and V opposite if one of them intersects the polar of the other, i.e. if $\det(U^T \Omega V) = 0$. Let Z be a certain infinitely distant element not opposite to the proper element A**.

* We note that in the Riemannian case any two elements have a common position.

** In the Riemannian case any Z and A are not opposite.

We normalize the elements A and Z by the conditions $A^T \Omega A = \mathcal{E}$, $Z^T \Omega A = \frac{1}{2}\mathcal{E}$. Then, as can be shown, all geodesic lines joining A and Z have the form

$$C_\lambda = Z e^{\lambda p} + (A - Z) e^{-\lambda p}, \tag{1}$$

where p is an arbitrary diagonal $(m+1) \times (m+1)$-matrix with positive elements, and λ is the geodesic parameter. Here $C_0 = A$, and the complex distance between the elements C_λ and A is equal to λp; moreover, $C_\lambda e^{-\lambda p} \to Z$ as $\lambda \to \infty$ (and this means that Z is an infinitely distant element of the geodesic (1)). For each element C_λ we construct the surface θ_λ obtained by all possible transformations of the element A by motions that leave fixed all points of the plane C_λ. The surface θ_λ will be called a **geodesic sphere**, and the plane C_λ its center. The equation of the geodesic sphere θ_λ has the form

$$X^T \theta_\lambda X = 0; \qquad \theta_\lambda = \Omega - \Omega C_\lambda [\mathcal{E} \operatorname{ch}^2 \lambda p]^{-1} A^T \Omega. \tag{2}$$

As $\lambda \to \infty$, the matrix θ_λ has the finite limit $\omega = \Omega - 4\Omega Z \mathcal{E} Z^T \Omega$, so that one may regard the geodesic sphere θ_λ in the limit as passing into the manifold of elements X satisfying the equation $X^T \omega X = 0$. We shall call this limiting manifold a **horosphere** in the space ${}^{l,k}S_{n,m}$, and the infinitely distant element Z the **directing element** of the horosphere. It is not hard to see that, in the case of Lobachevskii space ${}^{l}S_n = {}^{1,l}S_{n,0}$, this definition agrees with the classical one. Let us note that if the normalization condition on the matrix Z is dropped, then the horosphere will have the equation

$$X^T \omega X = 0, \qquad \omega = \Omega + \Omega Z H Z^T \Omega, \tag{3}$$

where H is an arbitrary nonsingular symmetric $(m+1) \times (m+1)$-matrix of index k.

From equation (3) it is easily derived that every element X of the horosphere (3) is nonopposite to Z_0, and that, conversely, through every nonopposite element Z of the space ${}^{l,k}S_{n,m}$ there passes a unique horosphere with directing element Z.

3. Our main task is to study the geometry prevailing on the horosphere ω. In the ambient space ${}^{l}S_{n}$ choose a basis so that

$$\Omega = \begin{bmatrix} & & I \\ & E & \\ I & & \end{bmatrix}, \qquad Z = \begin{bmatrix} I \\ \\ \end{bmatrix}, \tag{4}$$

where I is the identity $(m+1) \times (m+1)$-matrix, E is a diagonal matrix containing $n - l - m$ entries $+1$ and $l - m - 1$ entries -1, and zeros are understood in the empty positions.

We shall call the subgroup of the motion group of the space ${}^{l,k}S_{n,m}$ that maps ω into itself the **sliding group** $L(\omega)$ of the horosphere ω. It turns out that, in the basis (4), the sliding group $L(\omega)$ consists of motions with matrices

$$g = \begin{bmatrix} S & D & C \\ & U & F \\ & & T \end{bmatrix}, \qquad UEU^T = E; \qquad SHS^T = H; \qquad T^T = S^{-1}; \qquad D = -SF^T EU;$$

$$T^T C + C^T T + F^T E F = 0. \tag{5}$$

It also turns out that the group $L(\omega)$ has a metabelian radical $L_0(\omega)$, consisting of motions g_0 with matrices (5) in which $S = T = I$, $U = I$. Thus the matrix g_0 is uniquely determined by specifying the $(m+1) \times (n-2m-1)$-matrix $Q = D$ and the skew-symmetric $(m+1) \times (m+1)$-matrix $\Gamma = \frac{1}{2}(C - C^T)$. We shall call the radical $L_0(\omega)$ the **horospheri-**

subgroup of the group $L(\omega)$. It is not difficult to prove that the horospherical subgroup $L_0(\omega)$ acts simply transitively on the horosphere ω. This circumstance makes it possible to identify the horosphere ω with the group space of its horospherical subgroup. Thus, as coordinates of an element X of the horosphere ω one may take the matrices Q and Γ. It is not difficult to calculate that, if

$$X_1 = -{}^1/_2 H - {}^1/_2 QEQ^T + \Gamma; \qquad X_2 = -EQ^T, \tag{6}$$

then the matrix X consists of the following submatrices, one after another from top to bottom: X_1, X_2, I. Under the motions (5) from $L(\omega)$, the coordinates Q, Γ are transformed as follows:

$$\Gamma' = {}^1/_2(SQED^T - DEQ^{TS^{T)+S\Gamma}} S^T + {}^1/_2(CS^T - SC^T), \tag{7}$$

$$Q' = SQU^{-1} - DU^{-1}.$$

These formulas show that the group $L(\omega)$ is imprimitive and that the systems of imprimitivity are the manifolds $Q = \text{const}$, into which, therefore, the horosphere is fibered. We note that the base of the fibration, i.e. the manifold of systems of imprimitivity, can be interpreted as the manifold of planes of special dimension of the quasi-Euclidean space ${}^{k,l-m-1}S^n_{n-m-1}$ (see (4)). We also note that each manifold $Q = \text{const}$ can be interpreted as the set of skew-symmetric matrices Γ, in which the group of motions $\Gamma \to S\Gamma S^T + CS^T$ $(SHS^T = H)$ acts.

4. On the horosphere ω one can define a semi-Euclidean metric invariant with respect to the group $L(\omega)$,

$$d_0^2(Q_1, Q_2) = \operatorname{Tr}\{E(Q_1^T - Q_2^T)H(Q_1 - Q_2)\}. \tag{8}$$

Obviously, the metric d_0 degenerates on the manifolds $Q = \text{const}$. But for two elements belonging to one such manifold there exists another invariant

$$d_1^2(\Gamma_1, \Gamma_2) = \operatorname{Tr}\{(\Gamma_1 - \Gamma_2)H^{-1}(\Gamma_1 - \Gamma_2)^T H^{-1}\} = -\operatorname{Tr}\{(\Gamma_1 - \Gamma_2)H^{-1}\}^2. \tag{9}$$

The metric d_1 is invariant with respect to $L(\omega)$ only for those points for which $d_0 = 0$. We shall call the metric d_0 the principal metric, and d_1 the adjoined one. Thus, every pair of elements of the horosphere possesses an invariant d with respect to the sliding group $L(\omega)$. If these elements lie in different layers, then $d = d_0$; if they lie in one, then $d = d_1$.

5. I. M. Gelfand, M. I. Graev, and F. I. Karpelevich ([2,3]) gave a general definition of a horosphere in a symmetric Riemannian space. According to this definition, one must take a geodesic line leading to an infinitely distant element Z, and consider some one-parameter subgroup γ_λ of shifts along this geodesic (so that $\gamma_\lambda A \to Z$ as $\lambda \to \infty$). The **horospherical subgroup** is the set of motions g_0 satisfying the condition

$$\lim_{\lambda \to \infty} \gamma_\lambda^{-1} g_0 \gamma_\lambda = e, \tag{10}$$

where e is the identity of the group. A **horosphere** in a symmetric Riemannian space is a surface of transitivity of a horospherical subgroup.

It turns out that in the case of the Riemannian space ${}^{l,l}S_n,\ l{-}1$ our definition of a horosphere coincides with the definition of I. M. Gelfand, M. I. Graev, and F. I. Karpelevich. Moreover, horospheres in the pseudo-Riemannian spaces ${}^{l,k}S_{n,m}$ $(m < l)$ can also be defined by the method (10). To prove this it is enough to establish that the definition of the horospherical subgroup given in item 3 coincides with the definition by means of (10). If one uses the fact that every geodesic leading to infinity-

but a remote element Z and containing elements in general position not opposite to Z, has the form (1), then we find that every subgroup γ_λ in the basis (4) has the form

$$\gamma_\lambda = \begin{bmatrix} e^{\lambda p} & & \\ & U_\lambda & \\ & & e^{-\lambda p} \end{bmatrix}, \tag{11}$$

where p is the same matrix as in (1), and U_λ is a one-parameter subgroup of the group of pseudo-orthogonal matrices $(U_\lambda^T E U_\lambda = E)$. Using formula (11), it is no longer difficult to establish that equality (10) holds if and only if $g_0 \in L_0(\omega)$, whence it follows that $L_0(\omega)$ coincides with the horospherical subgroup in the sense of the definition of I. M. Gelfand, M. I. Graev, and F. I. Karpelevich.

6. In conclusion we shall construct a simple model of the manifold of all horospheres in the space ${}^{l,k}S_{n,m}$. We have seen that a horosphere ω is determined by specifying an infinitely remote element Z and a symmetric nonsingular matrix H of index k. Under a change of basis in Z, the matrix H transforms as the matrix of a quadratic form. We shall always agree to choose the basis in Z so that the matrix H coincides with $\mathcal{E}$. Thus the manifold of horospheres in ${}^{l,k}S_{n,m}$ can be interpreted as the set of $(n{+}1)\times(m{+}1)$-matrices Z of maximal rank, defined up to multiplication on the right by an arbitrary pseudo-orthogonal matrix V of index k, an $(m+1)\times(m+1)$-matrix, and satisfying the condition $Z^T\Omega Z = 0$, where Ω is the absolute matrix. In this model of the manifold of horospheres, motions act according to the formula $Z \to gZ$, where g is a motion matrix, i.e. an arbitrary nonsingular $(n{+}1)\times(n{+}1)$-matrix satisfying the condition $g^T\Omega g = \Omega$.

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1. B. A. Rosenfeld, *Uspekhi Mat. Nauk*, **13**, no. 6 (84), 21 (1958).

2. F. I. Karpelevich, *Dokl. Akad. Nauk SSSR*, **124**, 1199 (1959).
3. I. M. Gelfand, M. I. Graev, N. Ya. Vilenkin, *Integral Geometry and Related Questions of Representation Theory.* Moscow, 1962.
4. B. A. Rosenfeld, *Trudy Moskov. Mat. Obshch.*, **8**, 49 (1959).

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