MATHEMATICS

V. T. KHARIN

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. T. KHARIN

ON AN APPROXIMATE METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS FOR LINEAR EQUATIONS WITH ANALYTIC COEFFICIENTS

(Presented by Academician G. I. Petrov, 28 IV 1964)

1. Consider the boundary-value problem

$$L(\lambda)u = f, \tag{1}$$

where the following notation has been introduced: $f = f(x)$ is a function given on the interval $[x_1, x_2]$ of the real x-axis; $u = u(x)$ is the desired function, defined there as well; $L(\lambda)$ is a linear differential operator formed by a linear ordinary differential expression of order m with coefficients depending on the complex parameter λ, and also by homogeneous boundary conditions at the endpoints of the interval $[x_1, x_2]$.

We shall seek an approximate solution of problem (1) in the s-th approximation in the form

$$u^{(s)}(x) = \sum_{k=0}^{s-1} a_k^{(s)} u_k(x), \tag{2}$$

where $\{u_k(x)\}$ is a system of functions satisfying the boundary conditions. To find the constants $a_k^{(s)}$ $(k = 0, 1, \dots, s-1)$, choose a point $x_0 \in [x_1, x_2]$ and require that the relations

$$\left\{\frac{d^i}{dx^i}\left[L(\lambda)u^{(s)} - f\right]\right\}_{x=x_0} = 0 \quad (i = 0, 1, \dots, s-1), \tag{3}$$

be satisfied; these constitute a system of linear algebraic equations with respect to $a_k^{(s)}$ $(k = 0, 1, \dots, s-1)$.

By analogy with the well-known collocation method, we shall call equations (3) the **equations of the multiple-collocation method**.

2. Let us turn to the question of convergence of this method. We may assume without loss of generality that $x_0 = 0$. We shall denote by C_r and S_r, respectively, the boundary and the interior of the circle of radius $r > 0$ with center at the origin in the plane of the complex variable x.

Let $L(\lambda)$ and f in equation (1) satisfy the following requirements:

I. $L(\lambda) = M + N(\lambda)$, where M and $N(\lambda)$ are linear differential operators formed by ordinary linear differential expressions of orders m and n, respectively ($m > n$), and by homogeneous boundary conditions; moreover, the equation $Mu = 0$ has only the trivial solution.

II. The coefficients of the differential expressions corresponding to the operators M and $N(\lambda)$, as well as the function $f(x)$, admit analytic continuations from the interval $[x_1, x_2]$ to a domain E of the complex x-plane that contains in its interior the closed circle of radius

$$R > \max_{i=1,2} |x_i|.$$

III. The coefficients of the differential expression corresponding to the operator $N(\lambda)$ are analytic functions of λ in a domain D of the complex λ-plane.

IV. $u_k(x) = M^{-1}x^k$ $(k = 0, 1, \ldots)$.

Under these conditions we shall reduce problem (1) to an equation with a completely continuous operator in a certain Hilbert space of analytic functions, in which the method of multiple collocation coincides with the Galerkin method and sufficient conditions for the convergence of the latter are satisfied.

Lemma 1. Let $R > 0$. The set of functions $\varphi(x)$ analytic in S_R, for which there exists

$$\lim_{r \to R-0} \oint_{C_r} |\varphi(\zeta)|^2 \, d\theta \equiv P(\varphi) < \infty,$$

where $\zeta = re^{i\theta}$, becomes a complete Hilbert space (we shall denote it by $H_2(R)$) if the norm is introduced as follows:

$$\|\varphi\|_R = \frac{1}{2\pi} \sqrt{P(\varphi)}. \tag{4}$$

Proof. It is enough to consider the case $R = 1$. For any function $\varphi(x)$ satisfying the conditions of the lemma, we have [(1)]

$$\varphi = \sum_{k=0}^{\infty} \varphi_k x^k,$$

where

$$\varphi_k = \frac{1}{2\pi r^k} \oint_{C_r} \varphi(\zeta)\bar{\zeta}^k \, d\theta \qquad (r < 1, \ k = 0, 1, \ldots) \tag{5}$$

(the bar above denotes complex conjugation). Moreover,

$$\frac{1}{2\pi} \oint_{C_r} |\varphi(\zeta)|^2 \, d\theta = \sum_{k=0}^{\infty} |\varphi_k|^2 r^{2k}, \tag{6}$$

where the left-hand side of (6) is a nondecreasing continuous function for $0 \leq r \leq 1$ (if it is extended to $r = 1$ by continuity). Since the series on the right-hand side of (6) consists of positive terms continuous at $r = 1$, it follows that

$$\|\varphi\|_1^2 = \sum_{k=0}^{\infty} |\varphi_k|^2, \tag{7}$$

i.e., the correspondence between functions $\varphi \in H_2(1)$ and the sequences of coefficients of their Taylor series $\{\varphi_k\}$ defines an isometric isomorphism between $H_2(1)$ and the Hilbert space l_2, which proves the lemma.

Let $u(x)$ be a solution of problem (1) under conditions I-III. Denote $Mu = v$. By condition I the last equality is uniquely solvable, i.e. $u = M^{-1}v$, and problem (1) may be replaced by the equivalent problem

$$v - N(\lambda)M^{-1}v = f. \tag{8}$$

It follows from condition II that the function $f(x)$, and also any solution of problem (1) and problem (8), are analytic in E and, consequently, belong to $H_2(R)$; i.e., it is enough to consider equation (8) in $H_2(R)$. Without loss of generality, we may assume $|x_i| < 1$ $(i = 1, 2)$ and consider (8) in $H_2(1)$.

Lemma 2. The operator $N(\lambda)M^{-1}$ is completely continuous in $H_2(1)$.

Proof. We shall determine the structure of the operator $N(\lambda)M^{-1}$. Arguing in the same way as in the construction of the inverse operator by Green' s function method [(2)], using condition II and the fact that $N(\lambda)$ contains differentiation of order less than m, we obtain $N(\lambda)M^{-1} = T_1(\lambda) + T_2(\lambda)$, where

$$T_i(\lambda)v = \int_{x_i}^{x} K_i(\lambda, x, y)v(y) \, dy \qquad (i = 1, 2). \tag{9}$$

Here $x \in S_1$, and integration is performed along any curve $\Gamma_i \in S_1$ joining x_i to x. The kernels have the form

$$K_i(\lambda, x, y) = \sum_{j=0}^{m-1} \alpha_{ij}(x, \lambda)\beta_{ij}(y) \qquad (i = 1, 2), \tag{10}$$

where $\alpha_{ij}(x, \lambda)$ are analytic in x in E, in λ in D, and $\beta_{ij}(y)$ are analytic in E. Since

$$T_i(\lambda)v = \sum_{j=0}^{m-1} \alpha_{ij}(x, \lambda) \int_{x_i}^{x} \beta_{ij}(y)v(y)\, dy, \qquad v \in H_2(1), \quad i = 1, 2,$$

it is enough to prove the complete continuity of the operators

$$B_{ij}v = \int_{x_i}^{x} \beta_{ij}(y)v(y)\, dy$$

and the boundedness of the operators $A_{ij}v = a_{ij}(x, \lambda)v(x)$. The latter follows from the relations

$$\|A_{ij}v\|_1^2 = \frac{1}{2\pi} \lim_{r\to 1-0} \oint_{C_r} |a_{ij}(\xi, \lambda)v(\xi)|^2\, d\theta \leq$$

$$\leq M_{ij}^2(\lambda)\frac{1}{2\pi} \lim_{r\to 1-0} \oint_{C_r} |v(\xi)|^2\, d\theta = M_{ij}^2(\lambda)\|v\|_1^2,$$

where $M_{ij}^2(\lambda) = \max_x |a_{ij}(x, \lambda)|^2$ for $|x| \leq 1$. Since B_{ij} is a particular case of the operator

$$Bv = \int_a^x \beta(y)v(y)\, dy,$$

where $\beta(y)$ is analytic in E, $|a| < 1$, and integration is performed along any curve joining a and x in S_1, we shall prove the complete continuity of Bv. Let

$$\beta(y) = \sum_{k=0}^{\infty} \beta_k y^k, \qquad v(y) = \sum_{k=0}^{\infty} v_k y^k, \qquad Bv = \sum_{k=0}^{\infty} (Bv)_k y^k.$$

By virtue of the isometry of $H_2(1)$ and l_2, it is enough to prove the complete continuity of B as an operator in l_2. By the absolute and uniform convergence inside S_1 of the Taylor series, we have

$$Bv = \int_a^x \sum_{i,j=0}^{\infty} \beta_i v_j y^{i+j}\, dy = \sum_{i,j=0}^{\infty} \beta_i v_j \frac{x^{i+j+1} - a^{i+j+1}}{i+j+1} = \sum_{k=0}^{\infty} (Bv)_k x^k,$$

where

$$(Bv)_0 = -\sum_{i,j=0}^{\infty} \frac{\beta_i a^{i+j+1}}{i+j+1} v_j, \qquad (Bv)_k = \frac{1}{k} \sum_{j=0}^{k-1} \beta_{k-j-1} v_j \quad (k > 0).$$

Thus the matrix of the operator B, considered in l_2, has the form

$$\beta_{0j} = -\sum_{i=0}^{\infty} \frac{\beta_i a^{i+j+1}}{i+j+1},$$

$$\beta_{kj} = \frac{1}{k} \beta_{k-j-1} \quad (k > 0,\ j < k),$$

$$\beta_{kj} = 0 \quad (k > 0,\ j \geq k).$$

Since

$$|b_{0j}|^2 \leq 2\pi \|\beta\|_1^2 \sum_i \left| \frac{a^{i+j+1}}{i+j+1} \right|^2$$

and $|a| < 1$, it is obvious that

$$\sum_{k,j=0}^{\infty} |b_{kj}|^2 < \infty,$$

and B is completely continuous, as was required to prove.

Let us now note that equations (3) are the condition that the first s coefficients of the Taylor series of the function $v^{(s)} - N(\lambda) M^{-1} v^{(s)} - f$ vanish.

By virtue of (5) and the fact that the scalar product in $H_2(1)$ has the form

$$(\varphi, \psi) = \frac{1}{2\pi} \lim_{r \to 1-0} \oint_{C_r} \varphi(\xi) \overline{\psi(\xi)}\, d\theta,$$

one may write (3) in the form

$$(v^{(s)} - N(\lambda) M^{-1} v^{(s)} - f,\ x^i) = 0 \qquad (i = 0, 1, \dots, s-1). \tag{11}$$

Since from condition IV it follows that

$$v^{(s)} = \sum_{k=0}^{s-1} a_k^{(s)} x^k,$$

the system (11) is the Galerkin system for equation (8) in the space $H_2(1)$, in which the functions x^k form a basis. All convergence conditions for the Galerkin method are satisfied ([3], [4]). Thus, the following has been proved.

Theorem. Suppose conditions I–IV are satisfied. Then:

1) If problem (1), for fixed $\lambda \in D$, has a nontrivial solution for every function $f \not\equiv 0$, then, beginning with some s, the approximate solutions of this problem by the multiple-collocation method exist and are uniquely determined, for the given f, from equations (2), (3). Their sequence converges in $H_2(R)$ to the exact solution.

2) All eigenvalues of problem (1) with $f \equiv 0$ that belong to the domain D, and only they, can be obtained as limits of all possible sequences of approximate eigenvalues (eigenvalues of system (3) with $f \equiv 0$) as $s \to \infty$, belonging to the domain D.

3) From any sequence of approximate eigenfunctions belonging to approximate eigenvalues $\lambda_n \to \lambda_0 \in D$, one can select at least one subsequence convergent in $H_2(R)$; moreover, every such subsequence converges to an exact eigenfunction belonging to the eigenvalue λ_0.

Remark 1. Convergence in $H_2(R)$ entails uniform convergence inside S_R; thus the approximate solutions of problem (1) converge to the exact ones uniformly inside S_R, together with derivatives of arbitrary order.

Remark 2. If in condition IV the system of functions $\{x^k\}$ is replaced by some other basis $\{\varphi_k\}$ in $H_2(R)$, then equations (3) or (11) will already be a system of equations not of the Galerkin method, but of the Galerkin–Petrov method in $H_2(R)$, and for the conclusions of the theorem to remain valid one must satisfy condition (A) of N. I. Pol' skii ([4]), or some equivalent condition connecting $\{x^k\}$ and $\{\varphi_k\}$.

The author expresses gratitude to Academician G. I. Petrov and V. A. Medvedev for discussion of the work and useful advice.

Scientific Research Institute of Mechanics
Moscow State University
named after M. V. Lomonosov

Received
23 IV 1964

References

[1] A. I. Markushevich, *Theory of Analytic Functions*, Moscow, 1950.
[2] M. A. Naimark, *Linear Differential Operators*, Moscow, 1954.
[3] S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Moscow, 1957.
[4] N. I. Pol' skii, *Ukr. Math. Journal*, **7**, No. 1, 1955.

Note: Figure translations are in progress. See original paper for figures.