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Abstract

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ON THE RECOVERY OF AN ANALYTIC FUNCTION FROM THE VALUES OF ITS DERIVATIVES, GENERALIZED IN THE SENSE OF A. O. GELFOND, AT A POINT

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Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $a_0 = 1$, $a_k \neq 0$ $(k = 1, 2, 3, ...)$, $\varlimsup_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{r}$, $0 < r < \infty$, and $\varliminf_{k\to\infty} \sqrt[k]{|a_k|} > 0$, be a certain fixed analytic function; then an arbitrary function $F(z)$, analytic in a circle of finite radius with center at the point $z = 0$, can be represented in the form $F(z) = \sum_{k=0}^{\infty} b_k a_k z^k$, where $\varlimsup_{k\to\infty} \sqrt[k]{|b_k|} = \frac{1}{\beta}$, $\beta > 0$. The function

$$D_f^n F(z) \overset{\text{def}}{=} \sum_{k=0}^{\infty} b_{n+k} a_k z^k \qquad (n = 0, 1, 2, ...), \tag{1}$$

which is regular in the circle $|z| < \beta r$, will be called the **Gelfond derivative of n-th order generated by the function** $f(z)$.

Such generalized derivatives were first introduced in the work of A. O. Gelfond and A. F. Leont' ev $(^1)^*$, where the representation of entire functions satisfying certain functional equations was studied; these equations were interpreted as equations in generalized derivatives of infinite order. Subsequently, they were used there and in other directions $(^{2-5})$.

In the present paper we consider one variant of A. O. Gelfond' s problem on determining an analytic function from its elements, where the values of the successive Gelfond derivatives at one point are taken as the elements. The location of this point substantially affects the nature of the solution of the problem. In one case it follows directly from the results of I. F. Lokhin $(^6)$, namely:

Theorem 1. *Let an arbitrary sequence of complex numbers* $\{\mathcal{D}_n\}_{n=0}^{\infty}$ *be given, with* $1/\varlimsup_{n\to\infty} \sqrt[n]{|\mathcal{D}_n|} = \beta > 0$, *and let* z_0 *be a point satisfying the condition* $|z_0| < \beta r_1$, *where* $r_1 = \min\{\rho, r\}$ *and* ρ *is the distance from the origin to the nearest zero of the function* $f(z)$. *Then there exists a unique function* $F(z)$, *analytic in the circle* $|z| < \beta r$, *for which* $D_f^n F(z_0) = \mathcal{D}_n$ $(n = 0, 1, 2, ...)$. *In* *the circle $|z| < \beta r$ this function is represented by the generalized Lokhin power series:*

$$F(z) = \sum_{k=0}^{\infty} \mathcal{D}_k(z, z_0)_f^k,$$

where $\{(z, z_0)_f^k\}_{k=0}^{\infty}$ are the coefficients in the expansion

$$\varphi(\xi) = f(z\xi)/f(z_0\xi) = \sum_{n=0}^{\infty} (z, z_0)_f^n \xi^n.$$

The case $|z_0| > \beta r_1$ proves more complicated. One of its special features is that one can indicate functions all of whose values of the deriva–

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* A. F. Leont' ev informed the author that the idea of such a generalization of derivatives belongs to A. O. Gelfond.

Gelfond derivatives of which at such points consist only of zeros, namely: assuming henceforth, for simplicity, that $\beta \geqslant 1$ and that the point z_0 belongs to the domain $r_l < |z_0| < r_{l+1}$, where $r_j < r_{j+1}$, $r_j = |z_j|$, $f(z_j) = 0$, $k(j)$ is the multiplicity of the root z_j $(j = 1, 2, \dots, l, l+1)$; then the following is valid

Theorem 2.

$$F(z) = \sum_{j=1}^{l} \sum_{\nu=0}^{k(j)-1} c_{j\nu} \frac{d^\nu}{d\tau_j^\nu} [f(\tau_j z)]_{\tau_j = z_j/z_0},$$

where $c_{j\nu}$ are arbitrary constants, is the general form of a function analytic in the disk $|z| < r$, whose Gelfond derivatives of orders $n \geqslant 0$ at the point z_0 are equal to zero.

This theorem follows from a result of Perron (^{7}).

In order that every function $F(z)$ could be represented by a series whose coefficients are the values of the Gelfond derivatives, in the case $r_l < |z_0| < r_{l+1}$ it proves necessary to introduce also Gelfond derivatives of negative orders:

$$D_f^{-n} F(z) \overset{\text{def}}{=} \sum_{k=0}^{\infty} b_{k-n} a_{kz}^k \qquad (n = 1, 2, 3, \dots), \tag{2}$$

where $\{b_{-m}\}_{m=1}^{\infty}$, generally speaking, are arbitrary constants, having the meaning

$$b_m = D_f^m F(0).$$

From definitions (1) and (2) it follows that specifying $F(z)$ and the set $\{b_{-m}\}_{m=1}^{\infty}$ uniquely determines the Gelfond derivatives of any order n $(n = 0, \pm 1, \pm 2, \dots)$ at every point of the disk $|z| < r$.

The system of functions by means of which $F(z)$ is reconstructed is constructed analogously: in the circular annulus $r_l < |z| < r_{l+1}$ there is a representation

$$\frac{1}{f(z)} = \tilde f_l(z) + \sum_{j=1}^{l} \sum_{\nu=0}^{k(j)-1} \frac{A_{j,\nu+1}}{(z-z_j)^{\nu+1}},$$

where

$$\tilde f_l(z) = \sum_{k=0}^{\infty} \tilde a_k^{(l)} z^k$$

is regular in the disk $|z| < r_{l+1}$; $A_{j,k(j)}, \ldots, A_{j,1}$ are the coefficients of the Laurent expansion of $1/f(z)$ in a neighborhood of z_j. Let $|z| < r$ and $r_l < |z_0| < r_{l+1}$. Then the indicated system, which we shall denote by

$$(z,z_0)^n_{f;l} \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{3}$$

is formed by the coefficients in the expansion in powers of ζ of the function

$$\frac{f(z\zeta)}{f(z_0\zeta)} = \sum_{n=-\infty}^{\infty} (z,z_0)^n_{f;l}\, \zeta^n$$

in the domain

$$r_l < |z_0\zeta| < r_{l+1}.$$

For $n = 0, 1, 2, \ldots$

$$(z,z_0)^n_{f;l} = \sum_{k=0}^{n} a_{n-k}\, \tilde a_k^{(l)} z_0^k z^{n-k} + \sum_{j=1}^{l} \sum_{\nu=0}^{k(j)-1} \frac{A_{j,\nu+1}}{\nu!\, z_0^{\nu+1}} \frac{d^\nu}{d\tau_j^\nu} \left\{ \left[f(z\tau_j) - \sum_{k=0}^{n} a_k z_0^k \tau_j^k \right] \tau_j^{-n-1} \right\}_{\tau_j = z_j/z_0}, \tag{3'}$$

and for $m = -1, -2, -3, \ldots$ it will be

$$(z,z_0)^m_{f;l} = \sum_{j=1}^{l} \sum_{\nu=0}^{k(j)-1} \frac{A_{j,\nu+1}}{\nu!\, z_0^{\nu+1}} \frac{d^\nu}{d\tau_j^\nu} \left[\tau_j^{-m-1} f(z\tau_j) \right]_{\tau_j = z_j/z_0}. \tag{3''}$$

Let us note some properties of the system (3):

1) If $|z| \leqslant r - \varepsilon$, $r_l + \varepsilon_1 \leqslant |z_0| \leqslant r_{l+1} - \varepsilon_2$, where $\varepsilon, \varepsilon_1, \varepsilon_2$ are arbitrary small positive numbers, then there exist constants $0 < M$ and $0 < \eta < 1$, depending on $\varepsilon, \varepsilon_1, \varepsilon_2$, such that

$$|(z,z_0)^n_{f;l}| < M\eta^{|n|} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

2)

$$(z,z_0)^n_{f;l} = \sum_{m=0}^{\infty} (0,z_0)^{n-m}_{f;l} a_m z^m.$$

3) $D^m_f(z, z_0)^n_{f,l} = (z, z_0)^{n-m}_{f,l}$ $(m = 0, 1, 2, \ldots)$.

4) $(z_0, z_0)^0_{f,l} = 1$, $\quad (z_0 z_0)^n_{f,l} = 0$ $(n = \pm 1, \pm 2, \pm 3, \ldots)$.

A second peculiarity is connected with the properties of the system (3), namely that one has to consider expansions in functions among which there are linearly dependent ones. More precisely:

Theorem 3. In order that, in the disk $|z| < r$,

$$\sum_{n=-\infty}^{\infty} \mathcal{D}_{-m}(z, z_0)^n_{f,l} \equiv 0$$

($\mathcal{D}_{-m}$ are constants), it is necessary and sufficient that

$$\overline{\lim_{m\to\infty}} \sqrt[m]{|\mathcal{D}_{-m}|} \leqslant 1$$

and that the analytic function

$$\sum_{m=1}^{\infty} \mathcal{D}_{-m} \tau^{m-1}$$

have as its zeros $\tau_j = z_j/z_0$ $(j = 1, \ldots, l)$, and that the multiplicities of these zeros be equal to $k(j)$, respectively.

We now give the result concerning the representation of $F(z)$ mentioned above.

Theorem 4. Let

$$F(z) = \sum_{k=0}^{\infty} b_k a_k z^k, \quad \beta \geqslant 1, \quad r_l < |z_0| < r_{l+1}.$$

Then in the disk $|z| < r$ the representation

$$F(z) = \sum_{n=-\infty}^{\infty} D^n_f F(z_0) \cdot (z, z_0)^n_{f,l} \tag{4}$$

holds, where $D^{-m}_f F(z)$ $(m = 1, 2, 3, \ldots)$ are determined with the aid of any set of numbers $\{b_{-m}\}_{m=1}^{\infty}$ satisfying the condition

$$\overline{\lim_{m\to\infty}} \sqrt[m]{|b_{-m}|} \leqslant 1.$$

It follows from this theorem that the representation (4) is not unique.

In considering the main question we shall assume:

a) $f(z)$ has one more property: there exists

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/r.$$

This ensures that the class of functions analytic together with all their Gelfond derivatives of nonnegative orders in the disk $|z| < r$ coincides with the totality of all analytic functions in this disk. A necessary and sufficient condition for $F(z)$ to belong to the indicated class is

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} \leqslant 1.$$

b) z_0 satisfies the inequalities $r_l \leqslant |z_0| < r_{l+1}$. Constructing for such points z_0 by formulas $(3')$ and $(3'')$ the system $\{(z, z_0)^n_{f,l}\}$, we consider the class of functions $\{\mathfrak{F}(z)\}$ defined by the series

$$\mathfrak{F}(z) = \sum_{n=-\infty}^{\infty} \mathcal{D}_n (z, z_0)^n_{f,l}, \tag{5}$$

where the following restrictions are imposed on the coefficients $\mathcal{D}_n$:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\mathcal{D}_n|} \leqslant 1, \qquad \overline{\lim_{n\to-\infty}} \sqrt[|n|]{|\mathcal{D}_n|} < 1. \tag{6}$$

The series (5) converges uniformly in every disk $|z| \leqslant r - \varepsilon$ ($\varepsilon > 0$ arbitrarily small), which follows from (6) and a simple refinement of property 1) of the system (3). Thus, every $\mathfrak{F}(z)$ is an analytic function in the disk $|z| < r$ and, consequently, by the series (5) there is uniquely determined a set $\{b_k\}_{k=0}^{\infty}$ such that

$$\mathfrak{F}(z) = \sum_{k=0}^{\infty} b_k a_k z^k, \quad \overline{\lim_{k\to\infty}} \sqrt[k]{|b_k|} \leqslant 1.$$

In terms of the coefficients of the series (5), this set is expressed by the formulas

$$b_k = \sum_{n=-\infty}^{\infty} \mathcal{D}_{n+k} (0, z_0)^n_{f,l}$$

$(k = 0, 1, 2, \ldots)$. Now construct

$$b_{-m} = \sum_{n=-\infty}^{\infty} \mathcal{D}_{n-m} (0, z_0)^n_{f,l} \quad (m = 1, 2, 3, \ldots).$$

For such $\{b_{-m}\}_{m=1}^{\infty}$ we have $\overline{\lim\limits_{m\to\infty}} \sqrt[m]{|b_{-m}|} < 1$. Setting $D_f^{-m}\mathcal{F}(0) = b_{-m}$ $(m = 1, 2, 4, \ldots)$, we obtain $D_f^n \mathcal{F}(z_0) = \mathcal{D}_n$ $(n = 0, \pm 1, \pm 2, \ldots)$.

The question arises: do there exist other functions, distinct from $\mathcal{F}(z)$, analytic in a neighborhood of $z = 0$, for which $\{\mathcal{D}_n\}_{n=-\infty}^{\infty}$ are the values of the Gelfond derivatives at the point z_0? Such functions do exist, but the following holds.

Theorem 5. Let a point z_0 $(r_l \leqslant |z_0| < r_{l+1})$ and a sequence of complex numbers $\{\mathcal{D}_n\}_{n=-\infty}^{\infty}$ satisfying (6) be given. Then there exist a unique function $\mathcal{F}(z)$ and a set $\{b_{-m}\}_{m=1}^{\infty}$ for which the following conditions hold: 1) $\mathcal{F}(z)$ is regular in $|z| < r$; 2) $\overline{\lim\limits_{m\to\infty}} \sqrt[m]{|b_{-m}|} < 1$; 3) $D_f^n \mathcal{F}(z_0) = \mathcal{D}_n$ $(n = 0, 1, 2, ...)$; 4) for $D_f^{-m} \mathcal{F}(z)$ $(m = 1, 2, 3, ...)$, defined with the aid of $\{b_{-m}\}_{m=1}^{\infty}$, we have $D_f^{-m} \mathcal{F}(z_0) = D_{-m}$ $(m = 1, 2, 3, ...)$. $\mathcal{F}(z)$ and $\{b_{-m}\}$ are determined by the series (5).

Remark 1. Theorem 5 in fact establishes the equivalence of the sets of sequences of numbers $\{\mathcal{D}_n\}_{n=-\infty}^{\infty}$ and of sequences of analytic functions $\{D_f^n \mathcal{F}(z)\}_{n=-\infty}^{\infty}$.

Remark 2. Theorem 5 may also be regarded as a theorem on the existence, uniqueness, and representation of the solution of the infinite system

$$\sum_{k=0}^{\infty} a_k z_0^k b_{k+m} = D_m \qquad (m = 0, \pm 1, \pm 2, ...).$$

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Note: Figure translations are in progress. See original paper for figures.

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