MATHEMATICS

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Abstract

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MATHEMATICS

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SOLUTION OF THE CONJUGACY PROBLEM FOR ONE CLASS OF GROUPS COINCIDING WITH THEIR ANTICENTERS, BY MEANS OF A GENERALIZED DEHN ALGORITHM

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Let a group G be given by generators $a_1, \dots, a_n$ and defining relations $R_1 = 1, \dots, R_k = 1$, where: 1) each R_i is irreducible; 2) the set of words $\{R_i\}$ is closed under the operations of taking the inverse word and taking cyclic permutations of the letters of the words R_i; 3) if R_i and R_j are not mutually inverse, then, upon reduction of the product $R_{iR}j$, $< \frac{1}{6}$ of the letters of the word R_i are cancelled.

An arbitrary word from the set $\{R_i\}$ is called a defining word and is denoted by R; the symbol $\overline{T}$ denotes T^{-1}; graphical equality, equality in the free group, and equality in the group G are denoted respectively by $\simeq$, $\equiv$, and $=$; if $\varepsilon_i = \pm 1$, then $l(a_{i_1}^{\varepsilon_1} \dots a_{i_m}^{\varepsilon_m}) = m$–the length of the word $a_{i_1}^{\varepsilon_1} \dots a_{i_m}^{\varepsilon_m}$.

Theorem 1. *If an irreducible nonempty word $W = 1$ in the group G, then W contains, without overlap:*

1) $> \frac{5}{6}$ *of one R,*

or

2) $> \frac{4}{6}$ *of each of some two R,*

or

3) $> \frac{4}{6}$ *of one R and $> \frac{3}{6}$ of each of some two R,*

or

4) $> \frac{3}{6}$ *of each of some five R.*

This theorem strengthens the main theorem from ([6]), but for a narrower class of groups. Its proof is based on the weaker main theorem from ([5]) and Lemmas 1-4. Analogous results for similar classes of groups are set forth in ([1],[2],[4],[8],[10–12]).

Lemma 1. Let $R_i \simeq S\overline{T}$ and XSY be irreducible.

1) If the subword S is not contained in any longer common subword of the words SY and R, then TY is irreducible.

2) If $T \simeq T^{lT^r}$, $Y \simeq Y^{lY^r}$, and $R_j \simeq TY^{lV}$, then $l(T^l) < \frac{1}{6}l(R_i)$ and $l(T^r) < \frac{1}{6}l(R_j)$.

Lemma 2. Let $W \simeq XPQSY$, $R_i \simeq PQ\overline{T}$, and $R_j \simeq QS\overline{V}$.

1) If PQ is not contained in any longer common subword of the words W and R, then $l(Q) < \frac{1}{6}l(R_i)$ and $l(Q) < \frac{1}{6}l(R_j)$.

2) If $l(S) \geq \frac{1}{6}l(R_j)$ and $T \simeq Ua$, then aSZ is not a defining word.

Lemma 3. If XS_1S_2Y is irreducible, $R_i \simeq S_1\overline{T}_1$, $R_j \simeq S_2\overline{T}_2$, $T_1 \simeq A_1P$, and $T_2 \simeq \overline{P}A_2$, then $l(P) < \frac{1}{6}l(R_i)$ and $l(P) < \frac{1}{6}l(R_j)$.

If, moreover, $l(P) > 0$, $A_1 \simeq B_1C_1$, $A_2 \simeq B_2C_2$, and $R_k \simeq \overline{C}_1B_2Z$, then $l(C_1) < \frac{1}{6}l(R_k)$ and $l(B_2) < \frac{1}{6}l(R_k)$.

Lemma 4. If $R_i \cong S_1\overline{T}_1$, $R_j \cong S_2\overline{T}_2$, $T_1 \cong AB$, $S_2 \cong CD$, $R_k \cong BCZ$, $l(B) > 0$, and T_1T_2 is irreducible, then $l(C) < \frac{1}{6}l(R_k)$.

Theorem 2. If an irreducible nonempty circular word C is equal to 1 in the group G, then C either is a defining word written cyclically, or contains without intersection:

1) $> \frac{5}{6}$ of each of some two R,

or

2) $> \frac{4}{6}$ of each of some three R,

or

3) $> \frac{4}{6}$ of each of some two R and $> \frac{3}{6}$ of each of some two R,

or

4) $> \frac{4}{6}$ of one R and $> \frac{3}{6}$ of each of some five R,

or

5) $> \frac{3}{6}$ of each of some seven R.

This theorem strengthens Theorem 9 of ([6]). Its proof is based on Theorem 1 and Lemmas 1-3. The author has constructed examples showing that Theorems 1 and 2 cannot be strengthened by increasing the fractions or the integers in any of the cases.

Definition 1. Let

$$R_{i_1} \cong U_1V_1W_1, \qquad R_{i_2} \cong U_2\overline{W}_1V_2W_2, \qquad R_{i_3} \cong U_3\overline{W}_2V_3,$$

the words W_1 and W_2 being nonempty, $l(V_2) > \frac{4}{6}l(R_{i_2})$, and either $l(V_1) > \frac{2}{6}l(R_{i_1})$ and $l(V_3) > \frac{3}{6}l(R_{i_3})$, or $l(V_1) > \frac{3}{6}l(R_{i_1})$ and $l(V_3) > \frac{2}{6}l(R_{i_3})$. Then the word $V_1V_2V_3$ is called a **triple**, and V_2 is called its **core**.

Theorem 3. Let an irreducible nonempty circular word C be equal to 1 in the group G and satisfy the following conditions:

a) C is not a defining word written cyclically;

b) C does not contain without intersection:

1) $> \frac{5}{6}$ of each of some two R,

or

2) $> \frac{4}{6}$ of each of some three R,

or

3) $> \frac{3}{6}$ of each of some five R.

If, in addition, $R_i \cong S\overline{T}$, $\frac{5}{6}l(R_i) \geq l(S) > \frac{4}{6}l(R_i)$, and the subword S is contained in C, but is not contained in any longer common subword of the words R and C, then S is the core of a triple that is a subword of C.

In the proof of Theorem 3, Theorem 2 and Lemma 1 are used.

Definition 2. Let

$$R_{i_1} \cong U_1V_1W_1, \qquad R_{i_2} \cong U_2\overline{W}_1V_2W_2, \qquad R_{i_3} \cong U_3\overline{W}_2V_3W_3, \qquad R_{i_4} \cong U_4\overline{W}_3V_4,$$

the words W_1, W_2, W_3 being nonempty, $l(V_1) > \frac{2}{6}l(R_{i_1})$, $l(V_4) > \frac{2}{6}l(R_{i_4})$, and either

1) $l(V_2) > \frac{4}{6}l(R_{i_2})$ and $l(V_3) > \frac{3}{6}l(R_{i_3})$,

or

2) $l(V_2) > \frac{3}{6}l(R_{i_2})$ and $l(V_3) > \frac{4}{6}l(R_{i_3})$.

Then the word $V_1V_2V_3V_4$ is called a **quadruple**, and V_2 is its **core** in the first case, while V_3 is its **core** in the second.

Theorem 4. Let an irreducible nonempty circular word C be equal to 1 in the group G and contain without intersection:

1) $> \frac{5}{6}$ of one R,

or

2) $> \frac{4}{6}$ of each of some three R,

or

3) $> \frac{3}{6}$ of each of some five R.

If, in addition, $R_i \cong S\overline{T}$, $\frac{5}{6}l(R_i) \geq l(S) > \frac{4}{6}l(R_i)$, and S is contained in C, but is not contained in any longer common subword of the words R and C, then S is the core of a quadruple that is a subword of C.

The proof of Theorem 4 is based on Theorems 2 and 3 and Lemmas 1 and 3.

Theorem 5. If an irreducible nonempty circular word C is equal to 1 in the group G, then either C contains the words S_j $(i = 1, 2, 3, 4)$ without intersection, where

$$R_{i_j} \cong S_{jT}j, \qquad l(S_1) > \frac{4}{6}l(R_{i_1}), \qquad l(S_2) > \frac{4}{6}l(R_{i_2}), \qquad l(S_3) > \frac{3}{6}l(R_{i_3}),$$

$l(S_4) > \frac{3}{6}l(R_{i_4})$ and each of the words S_1 and S_2 is the core of a quadruple from C, or C contains, without intersection:

1) $> \frac{5}{6}$ of one R,

or

2) $> \frac{4}{6}$ of each of some three R' s,

or

3) $> \frac{3}{6}$ of each of some five R' s.

In the proof of Theorem 5, Theorems 2 and 4 are used. Theorem 5, in turn, is applied to obtain the following two theorems, which constitute the main results.

Theorem 6. *If $XY = YX$ in the group G, then X and Y are powers of one and the same element.*

This was proved for generators X in ([6]), then for all X conjugate to words every cyclic permutation of which is irreducible and does not contain $> \frac{1}{3}$ of a defining word, in ([9]), and for all elements of groups of one more restricted class in ([7]). The author has examples showing that the inequality $< \frac{1}{6}$ is the best possible for Theorem 6.

The conjugacy problem for the group G is solved in the following way. In order to determine whether any two given words X and Y are conjugate, i.e., whether there exists a word Z such that $X = \overline{Z}YZ$, write the word X on a circle; then reduce the resulting cyclic word and replace in it an occurrence of the word C by an occurrence of the word D, if $R_i \cong CD$ and $l(C) > l(D)$. We continue this process as long as it is possible. Splitting the cyclic word transformed in this way at all possible places, we obtain a sequence of words $A_1, \dots, A_r$. If $R_i \cong \overline{B}A_{jBC}j$, $l(C_j) < \frac{2}{6}l(R_i) < l(A_j)$, and all cyclic permutations $C_{j_1}, \dots, C_{j_s}$ of the word C_j are irreducible, then we replace the set $\{A_i\}$ by the set $\{C_{jk}\}$. Denote the words of the final list by $X_1, \dots, X_p$. From the word Y we obtain, in an analogous way, a list of words $Y_1, \dots, Y_q$. For each word Z such that

$$l(Z) < \frac{2}{3} \max_{1 \leq i \leq k} l(R_i),$$

and for each pair of words (X_i, Y_j), we determine whether the equality

$$X_i = \overline{Z} Y_{jZ}$$

holds.

Theorem 7. *The words X and Y are conjugate if and only if at least one of the finite number of equalities*

$$X_i = \overline{Z} Y_{jZ},$$

holds, where $1 \leq i \leq p$, $1 \leq j \leq q$, and

$$l(Z) < \frac{2}{3} \max_{1 \leq i \leq k} l(R_i).$$

This algorithm is a generalization of the algorithm which M. Dehn constructed for solving the conjugacy problem in another class of groups ([3]). Dehn's original algorithm was applied in ([6]) to a narrower class of groups than the class of groups considered in the present paper. Another generalization of Dehn' s algorithm was applied in ([8]) to a class of groups that does not contain the class of groups considered here and is not contained in it.

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Note: Figure translations are in progress. See original paper for figures.