ON THE REPRESENTATION OF EQUATIONS BY NOMOGRAMS OF THE SECOND KIND

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Abstract

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MATHEMATICS

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ON THE REPRESENTATION OF EQUATIONS BY NOMOGRAMS OF THE SECOND KIND

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Let

$$t_3 = f(t_1, t_2) \tag{1}$$

be a real equation with functions $f(t_1, t_2)$ sufficiently smooth in some domain G, and with the functions

$$M = -\frac{\partial f}{\partial t_2} : \frac{\partial f}{\partial t_1} \tag{2}$$

and the Saint-Robert function

$$W = \frac{\partial^2 \ln M}{\partial t_1 \, \partial t_2}. \tag{3}$$

Consider the question of representing equation (1) by a net nomogram with a rectilinear scale t_3. As is known, this leads to the search for a Massau equation

$$|\, f_{i1}(t_i);\ f_{i2}(t_i);\ f_{i3}(t_i)\,| = 0 \quad (i = 1, 2, 3), \tag{4}$$

which becomes an identity by virtue of (1).

We shall find formulas that determine, solely by means of quadratures, the functions f_{ik} of the anamorphosis (4). We shall show the uniqueness (up to collineation) of the representations (4) under consideration. We shall also indicate effective conditions for the nomographability (in the sense indicated above) of equation (1).

Similar questions related to the problem of general anamorphosis were considered after Gronwall ([1]) by many authors ([2–6]), etc.; analogous questions connected with the rectification of webs were considered in web geometry ([7]) by Bol, Libère, and others. Among the many important results obtained in these works there are no effective conditions and methods for obtaining the indicated representation (4); likewise, in the works pertaining to this subject ([8,9]), the uniqueness problem posed by Gronwall and partially solved in the present paper is not solved.

Without diminishing generality, we shall obviously assume that in equation (4)

$$f_{13} \equiv 1; \quad f_{23} \equiv 1; \quad f_{32} \equiv 1; \quad f_{33} \equiv 0.$$

We indicate a preliminary condition for nomographability.

Theorem 1. *Equation (1) is nomographable in the sense indicated above if and only if, for the given function M (2), there exists a solution with respect to f_{ik} of the Gronwall differential equation*

$$[f'_{21}(f_{12}-f_{22}) - f'_{22}(f_{11}-f_{21})] : [f'_{11}(f_{12}-f_{22}) - f'_{12}(f_{11}-f_{21})] = M, \quad (5)$$

where $f'_{ik} = df_{ik}/dt_i$. Moreover, among the solutions of equation (5) there necessarily exists a solution satisfying the following initial conditions:

$$\text{for } t_1 = t_{11},\ t_2 = t_{21} \qquad f_{1k} = 1, \quad f_{2k} = 0, \quad f'_{21} = 1, \quad f'_{22} = 0 \quad (k = 1, 2), \quad (6)$$

where $t_1 = t_{11},\ t_2 = t_{21}$ is any point of the domain G.

If equation (1) is nomographable, then, solving (4) with respect to f_{31}, we have

$$f_{31} = (f_{11} - f_{21}) : (f_{12} - f_{22}), \quad (7)$$

and, computing the function M (2), we arrive at equation (5). If, conversely, for the given function M (2) there exists a solution of equation (5), then the equation

$$\frac{\partial t_3}{\partial t_1}[f'_{21}(f_{12}-f_{22}) - f'_{22}(f_{11}-f_{21})] + \frac{\partial t_3}{\partial t_2}[f'_{11}(f_{12}-f_{22}) - f'_{12}(f_{11}-f_{21})] = 0. \quad (8)$$

It is not hard to see that its solution will be

$$t_3 = \Phi\left[(f_{11} - f_{21}) : (f_{12} - f_{22})\right] \quad (9)$$

or (7), if one sets $f_{31} = \Phi^{-1}$. Since (7) is representable in the form (4), the sufficiency of the condition has been proved.

If now equation (5) is solvable and

$$\left|\bar{f}_{i1}(t_i); \bar{f}_{i2}(t_i); \bar{f}_{i3}(t_i)\right| = 0 \tag{4'}$$

is some nomogram in the plane $\bar{x}\bar{y}$, then it is possible to construct a collineation, automorphic with respect to $\bar{x}_3 = 0$:

$$\bar{x} = a_{11}x + a_{21}y + a_{31}, \quad \bar{y} = a_{12}x + a_{22}y + a_{32}, \quad |a_{ik}| \neq 0, \tag{10}$$

which transforms $(4')$ into (4) with condition (6). It is easy to determine all a_{ik} and to find that the condition $|a_{ik}| = 0$ cannot occur for points from G. This proves the second part of the theorem.

Corollary. *If equation (1) is nomographable, then M (2) is representable in fractional-polynomial form (5) (i.e., as a ratio of two polynomial functions), where the dimension of the numerator and denominator does not exceed* 3 ([10]).

If the scale t_1 (or t_2) is rectilinear, then M (or $1/M$) is a function of dimension two. The solution of the problem in this case is known ([6]).

For brevity, introduce the following notation:

$$Q = M^2 W; \tag{11}$$

$$A = \frac{M^2}{4Q}\left[-M\left(\frac{Q}{M}\right)_1' - M^2\left(\frac{Q}{M^3}\right)_2'\right]; \tag{12}$$

$$C = \frac{M^2}{4Q}\left[\frac{Q^2}{M^3} - \left\{M^2\left(\frac{Q}{M^3}\right)_2'\right\}_1'\right]; \tag{13}$$

$$S = \frac{2AM_2'}{M} + \frac{3Q}{2} + A_1'M - AM_1' - A_2'; \tag{14}$$

$$R = \frac{2CM_2'}{M} - \frac{AQ}{M^2} + C_1'M - C_2' + M^2\left(\frac{Q}{M^3}\right)_2', \tag{15}$$

everywhere $W_{ik}'' = \partial^2 u/\partial t_i \partial t_k$.

Theorem 2. *A solution of equation (5) (if it exists) satisfying the initial conditions (6) is determined uniquely up to quadratures by means of the equation*

$$Sf_{12}' - \frac{S}{M}f_{22}' + R(f_{11} - f_{22}) = 0. \tag{16}$$

Equation (5), obviously, is equivalent to the system of equations

$$\frac{\partial f_{ik}}{\partial t_i} = \delta_i^j y_{ik} \quad (i, j, k = 1, 2), \tag{17}$$

where, as follows from equation (5),

$$y_{11} = \frac{1}{M}\left[y_{21} - y_{22}\frac{f_{11} - f_{21}}{f_{12} - f_{22}}\right] + \frac{y_{12}(f_{11} - f_{21})}{f_{12} - f_{22}}, \tag{18}$$

and δ_i^j is the Kronecker symbol.

If system (17) is completed to canonical form, containing on the left the derivatives of all unknown functions appearing on the right ([11]), with the aid of the condi-

conditions $\partial y_{11}/\partial t_2 = 0$, $\partial^2 y_{21}/\partial t_2 \partial t_1 = 0$, $\partial^2 y_{12}/\partial t_1 \partial t_2 = 0$, which determine these derivatives, then it will turn out that from the system obtained one can single out a subsystem containing only the unknown functions f_{12}, f_{22} and their derivatives up to and including the second order, namely, the subsystem

$$\frac{\partial f_{i2}}{\partial t_j} = \delta'_{ij} y_{i2}, \qquad \frac{\partial y_{i2}}{\partial t_j} = \delta'_{ij} z_{i2} \quad (i, j = 1, 2), \tag{19}$$

where z_{12} and z_{22} are expressed, respectively, as

$$z_{12} = y_{12}\left(\frac{A}{M^2} - \frac{M_1'}{M}\right) + \frac{1}{f_{12} - f_{22}}\left(\frac{3}{4}y_{12}^2 - \frac{1}{4M^2}y_{22}^2\right) - \\ - y_{22}\frac{A}{M^3} + (f_{12} - f_{22})\left(\frac{C}{M^2} - \frac{Q}{M^3}\right) - \frac{y_{12}y_{22}}{f_{12} - f_{22}}\frac{1}{2M}; \tag{20}$$

$$z_{22} = -y_{12}A + y_{22}\left(\frac{A}{M} + \frac{M_2'}{M}\right) + \frac{y_{12}^2}{f_{12} - f_{22}}\frac{M^2}{4} - \frac{3}{4}\frac{y_{22}^2}{f_{12} - f_{22}} - \\ -(f_{12} - f_{22})C + \frac{y_{12}y_{22}}{f_{12} - f_{22}}\frac{M}{2}. \tag{21}$$

System (19) is not completely integrable. From $\partial^2 y_{12}/\partial t_1 \partial t_2 = 0$ and $\partial^2 y_{22}/\partial t_2 \partial t_1 = 0$ we find that their linear combination gives equation (16).

Now, using the initial conditions (6), all the f_{ik} can be obtained uniquely by quadratures. Thus, from equation (16) we have

$$S^{0'} y_{12} + R^{0'} f_{12} = 0, \tag{22}$$

where $u^{0'} \equiv u(t_1, t_{21})$, whence f_{12} is found. Knowing f_{12}, from (16) we obtain f_{22}. Putting $t_2 = t_{21}$ in (5) and using (22), we find

$$y_{11} + \frac{R^{0'}}{S^{0'}} f_{11} = \frac{1}{M^{0'}}, \tag{23}$$

which determines f_{11}, and, finally, f_{21} is found from equation (5), in which all the remaining functions f_{ik} are already known.

Corollary. *All nomograms with a rectilinear scale t_3 (if they exist) of equation (1) are projective.*

The projectivity of nomograms of the second kind with a rectilinear scale t_1 of equation (1) was proved by S. V. Smirnov ([12]).

Remark. The formulas expressing successively the functions f_{ik} will be

$$f_{12} = \exp\left[-\int_{t_{11}}^{t_1} \frac{R^{0'}}{S^{0'}}\, dt_1\right];\tag{24}$$

$$f_{22} = \exp\left[-\int_{t_{21}}^{t_2} \frac{RM}{S}\, dt_2\right] \int_{t_{21}}^{t_2} \left[\frac{RM}{S} f_{12} + M f_{12}'\right] \exp\left[\int_{t_{21}}^{t_2} \frac{RM}{S}\, dt_2\right] dt_2;\tag{25}$$

$$f_{11} = \exp\left[-\int_{t_{11}}^{t_1} \frac{R^{0'}}{S^{0'}}\, dt_1\right] \left[\int_{t_{11}}^{t_1} \frac{1}{M^{0'}} \exp\left[\int_{t_{11}}^{t_1} \frac{R^{0'}}{S^{0'}}\, dt_1\right] dt_1 + 1\right];\tag{26}$$

$$f_{21} = \exp\left[-\int_{t_{21}}^{t_2} P\, dt_2\right] \int_{t_{21}}^{t_2} [P f_{11} + M f_{11}'] \exp\left[\int_{t_{21}}^{t_2} P\, dt_2\right] dt_2,\tag{27}$$

where $P = (f_{22}' - M f_{12}') : (f_{12} - f_{22})$.

The equations of the scale t_3 are easily found from the scales t_1, t_2 from equation (1). As for the conditions for nomographability of (1), they could be obtained in the following way.

Theorem 1 shows that equation (5), and consequently also system (19), admits a continuous 6-parameter group of transformations of the form (10). From the uniqueness of the representations (Theorem 2) it follows that this group is maximal. Therefore the integrability conditions of system (19), in the case of its compatibility, complete it to a completely integrable system containing 6 unknown functions (11). Since the system is compatible only for nomographable equations, these conditions give, in addition, the conditions for nomographability of equation (1) in the form of differential relations with M. However, after (24)–(27) these conditions can also be expressed as follows:

Theorem 3. *Equation (1) is nomographable with a rectilinear scale t_3 if and only if the functions f_{22} and f_{21}, defined by formulas (25) and (27), for the found functions f_{12} (24) and f_{11} (26), depend only on the variable t_2.*

From what has been set forth, for the geometry of two-dimensional webs ([7]) we have:

1) If a web with equation (1) and with curvature different from zero is topologically equivalent to a rectilinear web with a pencil of straight lines t_3, then the topological transformation that straightens the web in this way is unique up to collineations.

2. Formulas (24)–(27) determine only by quadratures the equations of the families of the rectilinear web (if it exists).

3. Theorem 3 indicates an effective condition for such straightening of a web.

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Note: Figure translations are in progress. See original paper for figures.