

# ON A MULTI-INDEX TRANSPORTATION PROBLEM WITH AXIAL SUMS

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1964

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**Abstract**

**Full Text**

**CYBERNETICS AND CONTROL THEORY**

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**ON A MULTI-INDEX TRANSPORTATION  
PROBLEM WITH AXIAL SUMS**

*(Presented by Academician A. I. Berg on 6 June 1963)*

We consider the problem of minimizing the linear form

$$\sum_{i_1, \dots, i_s} P_{i_1, \dots, i_s} x_{i_1, \dots, i_s} \tag{1}$$

under the conditions  $x_{i_1 \dots i_s} \geq 0$  and

$$\sum_{i_2, \dots, i_s} x_{i_1 \dots i_s} = a_{i_1}^{(1)}, \quad \sum_{i_1, \dots, i_{s-2}, i_s} x_{i_1 \dots i_s} = a_{i_{s-1}}^{(s-1)}, \quad \sum_{i_1, \dots, i_{s-1}} x_{i_1 \dots i_s} = a_{i_s}^{(s)}, \tag{2}$$

where, for all indicated  $i_k$ , summation is carried out from 1 to  $n_k$ ,  $i_l = 1, \dots, n_l$ ;  $l = 1, \dots, s$ ;  $0 \leq a_{i_l}^{(l)} < \infty$ . We shall call such a problem a problem with axial sums, or a problem  $T_{s-1}(s)$ . Multi-stage transportation problems, in particular, reduce to problems  $T_{s-1}(s)$ .

**Theorem 1.** *For a solution of the problem  $T_{s-1}(s)$  to exist, it is necessary and sufficient that the consistency conditions hold:*

$$\sum_{i_{l_1}=1}^{n_{l_1}} a_{i_{l_1}}^{(l_1)} = \sum_{i_{l_2}=1}^{n_{l_2}} a_{i_{l_2}}^{(l_2)} \quad \text{for all } 1 \leq l_1 < l_2 \leq s. \tag{3}$$

**Necessity** is obvious, since

$$\begin{aligned} \sum_{i_1} \left( \sum_{i_2, \dots, i_s} x_{i_1 \dots i_s} \right) &= \dots \\ \dots &= \sum_{i_s} \left( \sum_{i_1, \dots, i_{s-1}} x_{i_1 \dots i_s} \right). \end{aligned}$$

**Sufficiency.** By direct substitution it is easy to verify that

$$x_{i_1 \dots i_s} = T^{1-s} \prod_{l=1}^s a_{i_l}^{(l)}$$

satisfy all conditions (3), where

$$T = \sum_{i_l=1}^{n_l} a_{i_l}^{(l)}.$$

Consider the conditions

$$\sum_{i_2, \dots, i_s} x_{i_1 \dots i_s} = a_{i_1}^{(1)}, \dots, \sum_{i_1, \dots, i_{s-2}, i_s} x_{i_1 \dots i_s} = a_{i_{s-1}}^{(s-1)}. \quad (4)$$

Since

$$0 \leq x_{i_1 \dots i_s} \leq \min_{1 \leq l \leq s} a_{i_l}^{(l)} < \infty,$$

the conditions (4) generate, in the  $\prod_{l=1}^s n_l$ -dimensional space, a bounded convex polyhedron  $G$ . Denote by  $\xi^{(\sigma)}$  the extreme points of  $G$ ,  $\sigma = 1, \dots, h$ , where  $h$  is the number of all vertices of the polyhedron  $G$ . Then, if  $(x_{11\dots 1}, \dots, x_{n_1 n_2 \dots n_s}) \in G$ , then

$$x_{i_1 \dots i_s} = \sum_{\sigma=1}^h \xi_{i_1 \dots i_s}^{(\sigma)} \lambda_{\sigma}. \quad (5)$$

Here

$$\sum_{\sigma=1}^h \lambda_{\sigma} = 1, \quad \lambda_{\sigma} \geq 0. \quad (6)$$

Substituting (5) into (1) and into the  $s$ -th group of conditions (2), we obtain the following problem:

Find the minimum

$$\left( \sum_{\sigma=1}^h \gamma^{(\sigma)} \lambda_{\sigma} \right) \quad (7)$$

subject to

$$\sum_{\sigma=1}^h g_{i_s}^{(\sigma)} \lambda_{\sigma} = a_{i_s}^{(s)} \quad (8)$$

and (6). Here

$$\gamma^{(\sigma)} \equiv \sum_{i_1, \dots, i_s} p_{i_1 \dots i_s} \xi_{i_1 \dots i_s}^{(\sigma)}, \quad g_{i_s}^{(\sigma)} \equiv \sum_{i_1, \dots, i_{s-1}} \xi_{i_1 \dots i_s}^{(\sigma)}. \quad (9)$$

**Theorem 2.** *Conditions (6) follow from Theorem 1 and conditions (8).*

**Proof.** Summing (8) over all values of  $i_s$ , we obtain

$$\sum_{i_s} a_{i_s}^{(s)} = \sum_{i_s, \sigma} g_{i_s}^{(\sigma)} \lambda_{\sigma} = \sum_{\sigma} \lambda_{\sigma} \sum_{i_s} g_{i_s}^{(\sigma)} = T \sum_{\sigma} \lambda_{\sigma}.$$

Since  $\sum_{i_s} a_{i_s}^{(s)} = T$ , it follows that  $\sum_{\sigma} \lambda_{\sigma} = 1$ . The theorem is proved.

**Algorithm** for finding a basic feasible solution of the problem  $T_{s-1}(s)$ . Choose

$$x_{1\dots 1} = \min_{1 \leq l \leq s} a_1^{(l)}.$$

Find  $\bar{a}_1^{(l)} = a_1^{(l)} - x_{1\dots 1}$ . Note that at least one of the  $\bar{a}_1^{(l)}$  becomes zero. For definiteness, suppose that  $\bar{a}_1^{(1)} = 0$ . We proceed to finding

$$x_{21\dots 1} = \min(a_2^{(1)}, \bar{a}_2^{(2)}, \dots, \bar{a}_1^{(s)}).$$

In general, suppose that

$$x_{\tau_1 \dots \tau_s} = \min_{1 \leq l \leq s} (\tilde{a}_{\tau_l}^{(l)}) = \tilde{a}_{\tau_{l_*}}^{(l_*)}.$$

Here  $\tilde{a}_{\tau_l}^{(l)}$  denotes the current values. We proceed to finding

$$x_{\omega_1 \dots \omega_s} = \min_{1 \leq l \leq s} (\tilde{a}_{\omega_l}^{(l)}),$$

where

$$\omega_k = \begin{cases} \tau_k, & \text{if } k \neq l_*, \\ \tau_k + 1, & \text{if } k = l_*; \end{cases}$$

$$\tilde{a}_{\omega_k}^{(k)} = \begin{cases} \tilde{a}_{\tau_k}^{(k)} - x_{\tau_1 \dots \tau_s}, & \text{if } k \neq l_*, \\ a_{\omega_k}^{(k)}, & \text{if } k = l_*. \end{cases}$$

The process ends when  $\tau_k = n_k$  for all  $k = 1, \dots, s$ . From the algorithm for finding a basic feasible solution, two theorems follow:

**Theorem 3.** *If all  $a_{i_l}^{(l)}$  are integers, then the problem  $T_{s-1}(s)$  has at least one integer basic feasible solution.*

**Remark.** Unfortunately, an optimal integer solution does not always exist.

**Theorem 4.** *A basic feasible solution of the problem  $T_{s-1}(s)$  contains no more than*

$$\sum_{l=1}^s n_l - s + 1$$

*nonzero components.*

It can be shown that, when solving problem (7)–(8), in order to form the initial  $n_s$  columns  $g^{(1)}, \dots, g^{(n_s)}$ , one may use not only the extreme points of  $G$ , but also interior points.

Choose  $n_s$  points of  $G$ :

$$x_{i_1 \dots i_s}^{(j)} = \begin{cases} T^{2-s} \prod_{l=1}^{s-1} a_{i_l}^{(l)}, & \text{if } j = i_s, \\ 0, & \text{if } j \neq i_s. \end{cases}$$

Then

$$g_{i_s}^{(j)} = \begin{cases} T, & \text{if } j = i_s, \\ 0, & \text{if } j \neq i_s. \end{cases} \quad (10)$$

Substituting (10) into (8), we obtain  $\lambda_{i_s} = \frac{a_{i_s}^{(s)}}{T}$ ,

$$\gamma^{(j)} = \sum_{i_1, \dots, i_s} p_{i_1 \dots i_s} x_{i_1 \dots i_s}^{(j)} = T^{2-s} \sum_{i_1, \dots, i_{s-1}} p_{i_1 \dots i_{s-1} j} \prod_{l=1}^{s-1} a_{i_l}^{(l)}. \quad (11)$$

In view of (10), the matrix  $B$  of the initial  $n_s$  vectors  $g^{(1)}, \dots, g^{(n_s)}$  has the structure  $B = T \cdot E$ , where  $E$  is the identity matrix of order  $n_s$ ; hence the vector of estimates  $\bar{\pi} = (\pi_1, \dots, \pi_{n_s})$  is computed as follows:

$$\pi = \gamma_B B^{-1} = \gamma_B T^{-1} E, \quad \text{where } \gamma_B = (\gamma^{(1)}, \dots, \gamma^{(n_s)}).$$

In view of (11),

$$\pi_j = T^{1-s} \left( \sum_{i_1, \dots, i_{s-1}} p_{i_1 \dots i_{s-1} j} \prod_{l=1}^{s-1} a_{i_l}^{(l)} \right).$$

It follows from the duality theorem that the solution of problem (7)–(8) is optimal if the vector of estimates  $\bar{\pi}$  satisfies the inequalities

$$\gamma^{(\sigma)} \geq \bar{\pi} g^{(\sigma)}. \quad (12)$$

In expanded form, (12) is written as follows:

$$\sum_{i_1, \dots, i_s} (p_{i_1 \dots i_s} - \pi_{i_s}) \xi_{i_1 \dots i_s}^{(\sigma)} \geq 0, \quad \sigma = 1, \dots, h. \quad (13)$$

It is easy to see that, in order for (13) to hold, it is necessary and sufficient that

$$\min \sum_{i_1, \dots, i_s} (p_{i_1 \dots i_s} - \pi_{i_s}) \xi_{i_1 \dots i_s}^{(\sigma)} \geq 0, \quad \bar{\xi}^{(\sigma)} \in G. \quad (14)$$

Consider the following problem:

Find

$$\min \left[ \sum_{i_1, \dots, i_{s-1}} q_{i_1 \dots i_{s-1}} \eta_{i_1 \dots i_{s-1}}^{(\sigma)} \right] \quad (15)$$

subject to the conditions

$$\eta_{i_1 \dots i_{s-1}}^{(\sigma)} \geq 0, \quad \sum_{i_2, \dots, i_{s-1}} \eta_{i_1 \dots i_{s-1}}^{(\sigma)} = a_{i_1}^{(1)}, \dots, \quad \sum_{i_1, \dots, i_{s-2}} \eta_{i_1 \dots i_{s-1}}^{(\sigma)} = a_{i_{s-1}}^{(s-1)}. \quad (16)$$

Let

$$r \in R / \min_{1 \leq i_s \leq n_s} (P_{i_1 \dots i_s} - \pi_{i_s}) = P_{i_1 \dots i_{s-1} r} - \pi_r.$$

It is clear that  $R$  is, generally speaking, different for different sets of indices  $i_1, \dots, i_{s-1}$ . Denote

$$r^* = \min_{r \in R(i_1, \dots, i_{s-1})} r.$$

**Theorem 5.** Problem (15)–(16) is equivalent to problem (14) if

$$q_{i_1 \dots i_{s-1}} = p_{i_1 \dots i_{s-1} r^*} - \pi_{r^*},$$

and, moreover,

$$\xi_{i_1 \dots i_s}^{(\sigma_0)} = \begin{cases} \eta_{i_1 \dots i_{s-1}}^{(\sigma_0)}, & \text{if } i_s = r^*, \\ 0, & \text{if } i_s \neq r^*, \end{cases} \quad (17)$$

where  $\eta^{(\sigma_0)}$  is an optimal solution of problem (15)–(16).

For the proof, see (2).

Thus, the solution of problem  $T_{s-1}(s)$  has been reduced to the solution of problem  $T_{s-2}(s-1)$  (see (1), (2), (15), (16)). This procedure can be continued until we arrive at the solution of problem  $T_1(2)$ , i.e., the classical transportation problem. The solution  $T_1(2)$  can be found by the same method as indicated for the general case, but any other methods may also be applied.

other algorithms for solving the classical transportation problem. In particular, if one of the numbers  $n_1$  and  $n_2$  is significantly larger than the other, it is advisable to use the algorithm described above (1).

Let us return to solving problem (15), (16). If, as a result of the solution, it turns out that

$$\min \left[ \sum_{i_1, \dots, i_{s-1}} q_{i_1 \dots i_{s-1}} - \eta_{i_1 \dots i_{s-1}}^{(\sigma_0)} \right] < 0,$$

then by (17) we find all  $\xi_{i_1 \dots i_s}^{(\sigma_0)}$ , and then by (9) all  $g_{i_s}^{(\sigma_0)}$  and  $\gamma^{(\sigma_0)}$ , where  $\bar{g}^{(\sigma_0)}$  is the column to be introduced into the basis. In order to determine which column should be removed from the basis, and then the new basic solution and the new estimate vector, the modified simplex method is used. Thus, at each iteration of problem  $T_{s-1}(s)$ , one has to solve one problem  $T_{s-2}(s-1)$  and transform a matrix of size  $n_s \times (n_s + 1)$ .

In nondegenerate problems the basic solution contains exactly  $\sum_{l=1}^s n_l - s + 1$  positive components. From the algorithm for finding a basic solution it is clear that, in order for the problem to be nondegenerate, it is sufficient that for any  $1 \leq k < l \leq s$  there should not exist subsets  $I_k, I_l$  of the values of the indices  $i_k$  and  $i_l$  for which

$$\sum_{i_k \in I_k} a_{i_k}^{(k)} = \sum_{i_l \in I_l} a_{i_l}^{(l)}. \quad (18)$$

The nonfulfillment of (18) can be ensured by the method of small perturbations, considering  $\hat{a}_{i_l}^{(l)}$  instead of  $a_{i_l}^{(l)}$ , where

$$\hat{a}_{i_l}^{(l)} = \begin{cases} a_{i_l}^{(l)} + R_l, & i_l < n_l, \\ a_{n_l}^{(l)} + T_l, & i_l = n_l, \end{cases} \quad R_l = \begin{cases} \frac{R_{l+1}}{n_l}, & l = s-1, \dots, 2, \\ T_s = \varepsilon, & l = s, \\ 0, & l = 1. \end{cases} \quad (19)$$

All  $\hat{a}_{i_l}^{(l)}$  satisfy the compatibility conditions (3), whence

$$(n_l - 1)R_l + T_l = (n_k - 1)R_k + T_k \quad \text{for all } 1 \leq k < l \leq s. \quad (20)$$

From (19) it follows that

$$(n_k - 1)R_k < R_l \quad \text{for all } k < l. \quad (21)$$

From (20) and (21) it is easy to obtain that  $T_k > T_l + (n_l - 2)R_l$  for all  $k < l$ . It can be shown that for arbitrary  $I_k$  and  $I_l$

$$\sum_{i_l \in I_l} \hat{a}_{i_l}^{(l)} \neq \sum_{i_k \in I_k} \hat{a}_{i_k}^{(k)},$$

if  $\varepsilon < \frac{\delta}{n_s}$ , where

$$\delta = \min_{I_k, I_l} \left| \sum_{i_l \in I_l} a_{i_l}^{(l)} - \sum_{i_k \in I_k} a_{i_k}^{(k)} \right| > 0.$$

The author expresses gratitude to D. B. Yudin and E. G. Golshtein for valuable comments.

Received 6 VI 1963

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*Note: Figure translations are in progress. See original paper for figures.*

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