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Abstract

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SPACES OF FUNCTIONS ANALYTIC IN MULTIPLE HARTOGS DOMAINS

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In the present note, certain properties of multiple power series ([1]) are extended to multiple Hartogs series; with their aid an isomorphism is established for spaces of functions analytic in bounded multiple Hartogs domains. Then on these spaces a linear operator $L(f)$ is constructed, analogous to a linear functional on $E_R^{(n)}$ ([2]). The operator is applied to questions of basicity and completeness.

1. Let C^n be the space of n complex variables $z_1, \ldots, z_n$. A covering domain D over the space C^n having the property that, together with each point $z_0 = (z_1^{(0)}, \ldots, z_n^{(0)})$, it also contains all points $z = (z_1, \ldots, z_n)$ with coordinates

$$z_1 = (z_1^{(0)} - a_1)e^{i\theta} + a_1, \ldots,\ z_m = (z_m^{(0)} - a_m)e^{i\theta} + a_m;\ z_{m+1} = z_{m+1}^{(0)}, \ldots,\ z_n = z_n^{(0)},$$

where $0 \le \theta \le 2\pi$, will be called an m-fold Hartogs domain with planes of symmetry $z_1 = a_1, \ldots, z_m = a_m$. In the case $m = 1$ the domain D is called (([3], p. 109) a Hartogs domain. In the case of two variables a Hartogs domain is also called a semicircular domain (([4], p. 98; ([5], p. 228).

If, together with each point z_0 of the multiple Hartogs domain D, it also contains all points of the closed polycylinder

$$\{\, |z_j - a_j| \le |z_j^{(0)} - a_j|,\ j = 1, \ldots, m;\ z_{m+1} = z_{m+1}^{(0)}, \ldots,\ z_n = z_n^{(0)} \,\},$$

then the domain D is called complete.

To simplify notation we introduce the designations:

$$w = (z_1, \ldots, z_m), \quad z = (z_{m+1}, \ldots, z_n), \quad k = (k_1, \ldots, k_m), \quad \|k\| = k_1 + \cdots + k_m,$$

$$w^k = z_1^{k_1} \cdots z_m^{k_m}, \quad (w - a)^k = (z_1 - a_1)^{k_1} \cdots (z_m - a_m)^{k_m}.$$

Analogously to the case of a function of two variables (([4], p. 100), it is established that the domain of convergence of the m-fold Hartogs series

$$f(w,z)=\sum_{k=0}^{\infty} f_k(z)(w-a)^k \tag{1}$$

is a complete m-fold Hartogs domain with planes of symmetry $z_1=a_1,\dots,z_m=a_m$.

In what follows we shall place the origin of coordinates in the planes of symmetry of the Hartogs domain D, i.e., consider a complete m-fold Hartogs domain D with planes of symmetry $z_1=0,\dots,z_m=0$, which is the domain of convergence of the m-fold Hartogs series

$$f(w,z)=\sum_{k=0}^{\infty} f_k(z)w^k. \tag{2}$$

The coordinates of all points of the domain D are determined by the inequalities $|z_1|<R_1(z),\dots,|z_m|<R_m(z)$, where $R_1(z),\dots,R_m(z)$ are nonnegative functions defined in the projection H_D of the domain D onto the space of the variables z. Therefore the complete multiple Hartogs domain D with planes of symmetry $z_1=0,\dots,z_m=0$ will be denoted as follows:

$$D\{\,|z_j|<R_j(z),\ j=1,\dots,m;\ z\in H_D\,\}.$$

The functions $f_k(z)$ are regular in the domain H_D and satisfy the condition

$$\varlimsup_{z'\to z}\ \varlimsup_{\|k\|\to\infty}\sqrt[\|k\|]{|f_k(z')|\,R_1^{k_1}(z)\cdots R_m^{k_m}(z)}\leqslant 1.$$

2. Analogously to (1), let us introduce into consideration the sequence

$$d_k(D)=\sup_{(w,z)\in D}|w^k|,\qquad k_j=0,1,\dots;\ j=1,\dots,m.$$

Theorem 1. *If the function $f(w,z)$ is regular in the closed domain $\overline{D}$ and*

$$f(w,z)=\sum_{k=0}^{\infty} f_k(z)\,w^k,$$

then

$$|f_k(z)|\leqslant\frac{\max_{(w,z)\in\overline{D}}|f(w,z)|}{d_k(D)}. \tag{3}$$

The theorem is proved analogously to Theorem 3.8 from ([5]), p. 62.

Theorem 2. *If the series (2) converges in the bounded complete multiple Hartogs domain D, then for every subdomain D_0 of it, $\overline{D}_0 \subset D$, the series*

$$\sum_{k=0}^{\infty} \sup_{D_0} |f_k(z)|\, d_k(D_0) \tag{4}$$

converges.

Proof. Consider the domain $D_r\{(w,z) : (\frac{w}{r}, z) \in D\}$, where $0 < r < \infty$. Choose numbers r_1 and r_2, $0 < r_1 < r_2 < 1$, so that $\overline{D}_0 \subset D_{r_1} \subset D_{r_2} \subset D$. Since $d_k(D_{r_1}) = r_1^{\|k\|} d_k(D)$, $d_k(D_{r_2}) = r_2^{\|k\|} d_k(D)$, we have

$$\sum_{k=0}^{\infty} \sup_{D_0} |f_k(z)|\, d_k(D_0) \leqslant \sum_{k=0}^{\infty} \sup_{D_{r_1}} |f_k(z)|\, d_k(D_{r_1}) \leqslant$$

$$\leqslant M \sum_{k=0}^{\infty} \frac{d_k(D_{r_1})}{d_k(D_{r_2})} = M \sum_{k=0}^{\infty} \left(\frac{r_1}{r_2}\right)^{\|k\|} = M \sum_{j=0}^{\infty} \frac{(n+j-1)!}{j!(n-1)!} \left(\frac{r_1}{r_2}\right)^{j} < \infty,$$

where $M = \max_{\overline{D}_{r_2}} |f(w,z)|$.

Corollary. *If the series (2) converges in the bounded complete multiple Hartogs domain D, then in every subdomain D_0 of it, $\overline{D}_0 \subset D$, the series*

$$\sum_{k=0}^{\infty} f_k(z) d_k(D_0)$$

converges uniformly.

Theorem 3. *For convergence of the series (2) in the bounded complete multiple Hartogs domain*

$$D\{\, |z_1| < R_1(z), \dots, |z_m| < R_m(z);\ z \in H_D \,\}$$

it is necessary and sufficient that the series

$$\sum_{k=0}^{\infty} \sup_{z \in H_D} |f_k(z)|\, d_k(D)\, w^k \tag{5}$$

converge in the domain $D^{(1)}\{\, |z_1| < 1, \dots, |z_m| < 1;\ z \in H_D \,\}$.

Proof. Consider the domain D_r, $0 < r < 1$. Suppose that the series (5) converges in the domain $D^{(1)}$. This means that, for every $r < 1$, the series

$$\sum_{k=0}^{\infty} \sup_{D_r} |f_k(z)|\, d_k(D)\, r^{\|k\|}$$

converges, or the series

$$\sum_{k=0}^{\infty} \sup_{D_r} |f_k(z)|\, d_k(D_r).$$

converges. But

$$|f_k(z) w^k| \leq \sup_{D_r} |f_k(z)|\, d_k(D_r),$$

and therefore the series (2) converges in the domain D_r for every $r < 1$, and consequently it also converges in the domain D.

The converse assertion follows from Theorem 2.

Analogously to Theorem 3 one proves

Theorem 4. *For the convergence of the series (2) in the domain D it is necessary and sufficient that the series*

$$\sum_{k=0}^{\infty} f_k(z) d_k(D) w^k$$

converge in the domain $D^{(1)}$.

3. Consider the space $A(D)$ of functions analytic in the complete multiple Hartogs domain D, with the topology determined by uniform convergence of the functions $f(w,z) \in A(D)$ in each domain D_0, $D_0 \Subset D$.

Introduce the countably normed space $B(D)$ of sequences of functions $a = \{f_k(z)\}$ with the system of norms

$$\|a\|_r = \sum_{k=0}^{\infty} \sup_{z \in H_D} |f_k(z)|\, d_k(D)\, r^{\|k\|}, \qquad 0 < r < 1.$$

Next consider the space $A(\overline{D})$ of functions analytic in the closed domain $\overline{D}$, and the countably normed space $B(\overline{D})$ of sequences of functions $a = \{f_k(z)\}$ such that, for some $r > 1$, depending, generally speaking, on the sequence,

$$\|a\|_r = \sum_{k=0}^{\infty} \sup_{z\in H_D} |f_k(z)|\, d_k(D)\, r^{\|k\|} < \infty.$$

The topologies of $A(\overline{D})$ and $B(\overline{D})$ are defined analogously to ([1]).

Theorem 5. *If D is a bounded complete multiple Hartogs domain, then the spaces $A(D)$ and $B(D)$, and also $A(\overline{D})$ and $B(\overline{D})$, are isomorphic.*

Proof. Theorems 3 and 4 put each function $f(w,z) \in A(D)$ into correspondence with a sequence of functions $\{f_k(z)\}$, and conversely. This correspondence is continuous in both directions with respect to the topologies of the spaces $A(D)$ and $B(D)$, since for any numbers r and r_1, $0<r<r_1<1$, the inequalities

$$\max_{D_r} |f(w,z)| \le \sum_{k=0}^{\infty} \sup_{D_r} |f_k(z)|\, d_k(D_r) = \sum_{k=0}^{\infty} \sup_{D_r} |f_k(z)|\, d_k(D)\, r^{\|k\|} = \|a\|_r,$$

$$\|a\|_r \le \sum_{k=0}^{\infty} \frac{d_k(D) r^{\|k\|}}{d_k(D_{r_1})} \max_{D_{r_1}} |f(w,z)| = \left(\max_{D_{r_1}} |f(w,z)|\right) \sum_{k=0}^{\infty} \left(\frac{r}{r_1}\right)^{\|k\|}.$$

are valid.

The isomorphism of the spaces $A(\overline{D})$ and $B(\overline{D})$ is proved analogously.

Theorem 6. *If $D\{|z_j| < R_j(z),\ j=1,\dots,m;\ z\in H_D\}$ and $D_1\{|z_j| < R_j^{(1)}(z),\ j=1,\dots,m;\ z\in H_{D_1}\}$ are arbitrary bounded complete multiple Hartogs domains having the same projections $H_D = H_{D_1}$, then the spaces $A(D)$ and $A(D_1)$ are isomorphic. The spaces $A(\overline{D})$ and $A(\overline{D_1})$ are also isomorphic.*

Proof. By Theorem 3, $B(D)$ is isomorphic to $B(D^{(1)})$. Hence, from Theorem 5, it follows that $A(D)$ is isomorphic to $A(D^{(1)})$. Similarly, $A(D_1)$ is isomorphic to $A(D^{(1)})$. Consequently, $A(D)$ and $A(D_1)$ are isomorphic.

The spaces $A(\overline{D})$ and $A(\overline{D_1})$ are considered analogously.

Remark. If the domain D is univalent, then the space $A(D)$ is isomorphic to the space $A(E_1)$, where $E_1\{|z_1|<1,\dots,|z_m|<1\}$, which follows as a special case also from Theorem 2 ([6]).

4. Theorem 7. *In order that an operator $L(f)$, defined on $A(D)$ and taking values in $A(H_D)$, be linear and continuous, it is necessary*

and it is sufficient that

$$L(f) = \sum_{k=0}^{\infty} l_k \frac{1}{k!} \frac{\partial^k f(0,z)}{\partial w^k}, \tag{6}$$

where

$$\overline{\lim_{|k|\to\infty}} \sqrt[|k|]{|l_k|} = l < \overline{\lim_{|k|\to\infty}} \sqrt[|k|]{a_k(D)} \left(k! = k_1! \cdots k_m!, \quad \frac{\partial^k}{\partial w^k} = \frac{\partial^{k_1+\cdots+k_m}}{\partial z_1^{k_1} \cdots \partial z_m^{k_m}} \right).$$

Consider the function

$$\varphi(w) = \sum_{k=0}^{\infty} \frac{l_k}{w^{k+1}}.$$

It is analytic in the domain $\{|z_j| > l,\ j = 1, \dots, m\}$, and moreover $\varphi(w) = 0$ if, for some j, $1 \le j \le m$, $z_j = \infty$. With the aid of this function the operator $L(f)$ can be represented in the form

$$L(f) = \frac{1}{(2\pi i)^m} \int_{|z_1|=r_1} \cdots \int_{|z_m|=r_m} f(w,z)\varphi(w)\,dw,$$

where $l < r_j < R_j(z)$, $j = 1, \dots, m$.

Theorem 8. In order that the system

$$\left\{ f_l(w,z) = \sum_{k=0}^{\infty} f_{k,l}(z) w^k \right\} \tag{7}$$

be a basis in $A(D)$, it is necessary and sufficient that the system

$$\{\varphi_l(z) = L(f_l(w,z))\} \tag{8}$$

be a basis in $A(H_D)$.

Theorem 9. In order that the system (7) be complete in $A(D)$, it is necessary and sufficient that the system (8) be complete in $A(H_D)$.

With the aid of Theorems 8 and 9, theorems analogous to Theorems 4–11, established in [7] for the case $m = 1$, are proved; moreover, the conditions of these theorems can now be formulated as necessary and sufficient.

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Note: Figure translations are in progress. See original paper for figures.